

M. Postnikov

# Lectures

in Geometry

SEMESTER II

*Linear*

*Algebra*

*and*

*Differential*

*Geometry*

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This textbook directly continues the first volume of a course of geometry (M. M. Postnikov. *Lectures in Geometry: Semester 1. Analytic Geometry*. Moscow, Mir Publishers, 1981) based on lectures read by the author at Moscow University for students specializing in mathematics. It contains 27 lectures, each a nearly exact reproduction of an original lecture. It treats linear algebra, with elementary differential geometry of curves and surfaces in three-dimensional space added to pave the way for further discussions.







**М. М. ПОСТНИКОВ**

**ЛЕКЦИИ ПО ГЕОМЕТРИИ**

**СЕМЕСТР 2**

**ЛИНЕЙНАЯ АЛГЕБРА**

**И**

**ДИФФЕРЕНЦИАЛЬНАЯ ГЕОМЕТРИЯ**

**МОСКВА «НАУКА»**

**Главная редакция физико-математической  
литературы**

M. POSTNIKOV

# LECTURES IN GEOMETRY

SEMESTER II

## LINEAR ALGEBRA AND DIFFERENTIAL GEOMETRY

*Translated from the Russian  
by Vladimir Shokurov*

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## PREFACE

This book is a direct continuation of the author's previous book\* and is akin to it in being a nearly faithful record of the lectures delivered by the author in the second semester of the first year at the Mathematics-Mechanics Faculty of Moscow State University named after M. V. Lomonosov to mathematical students (a course in *Linear Algebra and Analytic Geometry*). Naturally, in the selection of the material and the order of presentation the author was guided by the same considerations as in the first semester (see the Preface in [1]). The number of lectures in the book is explained by the fact that although the curriculum assigns 32 lectures to the course, in practice it is impossible to deliver more than 27 lectures.

The course in Linear Algebra and Analytic Geometry is just a part of a single two-year course in geometry, and much in this book is accounted for, as regards the choice of the material and its accentuation, by orientation to the second year devoted to the differential geometry of manifolds. In particular, it has proved possible (although it is not envisaged by the curriculum) to transfer part of the propaedeutic material of the third semester (the elementary differential geometry of curves and surfaces in three-dimensional space) to the second-semester course and this has

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\* M. M. Postnikov. *Lectures in Geometry: Semester 1. Analytic Geometry*. Moscow, Nauka Publishers, 1979 (English translation, Mir Publishers, Moscow, 1981, referred to as 1 in what follows).

substantially facilitated (not only for the lecturer but, what is of course more important, also for the students) the third semester course. At the same time, as experience has shown, this material appeals to the students and they learn it well on the whole already in the second semester.

*M. M. Postnikov*

October 27, 1977

# CONTENTS

Preface	5
Lecture 1	11
Vector spaces. Subspaces. Intersection of subspaces. Linear spans. A sum of subspaces. The dimension of a subspace. The dimension of a sum of subspaces. The dimension of a linear span	
Lecture 2	19
Matrix rank theorem. The rank of a matrix product. The Kronecker-Capelli theorem. Solution of systems of linear equations	
Lecture 3	28
Direct sums of subspaces. Decomposition of a space as a direct sum of subspaces. Factor spaces. Homomorphisms of vector spaces. Direct sums of spaces	
Lecture 4	36
The conjugate space. Dual spaces. A second conjugate space. The transformation of a conjugate basis and of the coordinates of covectors. Annulets. The space of solutions of a system of homogeneous linear equations	
Lecture 5	47
An annulet of an annulet and annulets of direct summands. Bilinear functionals and bilinear forms. Bilinear functionals in a conjugate space. Mixed bilinear functionals. Tensors	
Lecture 6	58
Multiplication of tensors. The basis of a space of tensors. Contraction of tensors. The rank space of a multilinear functional	

Lecture 7	64
The rank of a multilinear functional. Functionals and permutations. Alternation	
Lecture 8	72
Skew-symmetric multilinear functionals. External multiplication. Grassman algebra. External sums of covectors. Expansion of skew-symmetric functionals with respect to the external products of covectors of a basis	
Lecture 9	82
The basis of a space of skew-symmetric functionals. Formulas for the transformation of the basis of that space. Multivectors. The external rank of a skew-symmetric functional. Multivector rank theorem. Conditions for the equality of multivectors	
Lecture 10	92
Cartan's divisibility theorem. Plücker relations. The Plücker coordinates of subspaces. Planes in an affine space. Planes in a projective space and their coordinates	
Lecture 11	106
Symmetric and skew-symmetric bilinear functionals. A matrix of symmetric bilinear functionals. The rank of a bilinear functional. Quadratic functionals and quadratic forms. Lagrange theorem	
Lecture 12	118
Jacobi theorem. Quadratic forms over the fields of complex and real numbers. The law of inertia. Positively definite quadratic functionals and forms	
Lecture 13	127
Second degree hypersurfaces of an $n$ -dimensional projective space. Second degree hypersurfaces in a complex and a real-complex projective space. Second degree hypersurfaces of an $n$ -dimensional affine space. Second degree hypersurfaces in a complex and a real-complex affine space	
Lecture 14	140
The algebra of linear operators. Operators and mixed bilinear functionals. Linear operators and matrices. Invertible	

operators. The adjoint operator. The Fredholm alternative. Invariant subspaces and induced operators

## Lecture 15 151

Eigenvalues. Characteristic roots. Diagonalizable operators. Operators with simple spectrum. The existence of a basis in which the matrix of an operator is triangular. Nilpotent operators

## Lecture 16 160

Decomposition of a nilpotent operator as a direct sum of cyclic operators. Root subspaces. Normal Jordan form. The Hamilton-Cayley theorem

## Lecture 17 170

Complexification of a linear operator. Proper subspaces belonging to characteristic roots. Operators whose complexification is diagonalizable

## Lecture 18 179

Euclidean and unitary spaces. Orthogonal complements. The identification of vectors and covectors. Annulets and orthogonal complements. Bilinear functionals and linear operators. Elimination of arbitrariness in the identification of tensors of different types. The metric tensor. Lowering and raising of indices

## Lecture 19 191

Adjoint operators. Self-adjoint operators. Skew-symmetric and skew-Hermitian operators. Analogy between Hermitian operators and real numbers. Spectral properties of self-adjoint operators. The orthogonal diagonalizability of self-adjoint operators

## Lecture 20 199

Bringing quadratic forms into canonical form by orthogonal transformation of variables. Second degree hypersurfaces in a Euclidean point space. The minimax property of eigenvalues of self-adjoint operators. Orthogonally diagonalizable operators

## Lecture 21 208

Positive operators. Isometric operators. Unitary matrices. Polar factorization of invertible operators. A geometrical interpretation of polar factorization. Parallel translations



and centroaffine transformations. Bringing a unitary operator into diagonal form. A rotation of an  $n$ -dimensional Euclidean space as a composition of rotations in two-dimensional planes

Lecture 22	221
Smooth functions. Smooth hypersurfaces. Gradient. Derivatives with respect to a vector. Vector fields. Singular points of a vector field. A module of vector fields. Potential and irrotational vector fields. The rotation of a vector field. The divergence of a vector field. Vector analysis. Hamilton's symbolic vector. Formulas for products. Compositions of operators	
Lecture 23	243
Continuous, smooth, and regular curves. Equivalent curves. Regular curves in the plane and graphs of functions. The tangential hyperplane of a hypersurface. The length of a curve. Curves in the plane. Curves in a three-dimensional space	
Lecture 24	262
Projections of a curve onto the coordinate planes of the moving $n$ -hedron. Frenet's formulas for a curve in the $n$ -dimensional space. Representation of a curve by its curvatures. Regular surfaces. Examples of surfaces	
Lecture 25	276
Vectors tangential to a surface. The tangential plane. The first quadratic form of a surface. Mensuration of lengths and angles on a surface. Diffeomorphisms of surfaces. Isometries and the intrinsic geometry of a surface. Examples. Developables	
Lecture 26	291
The tangential plane and the normal vector. The curvature of a normal section. The second quadratic form of a surface. The indicatrix of Dupin. Principal curvatures. The second quadratic form of a graph. Ruled surfaces of zero curvature. Surfaces of revolution	
Lecture 27	310
Weingarten's derivation formulas. Coefficients of connection. The Gauss theorem. The necessary and sufficient conditions of isometry	
Subject Index	346

# Lecture 1

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*Vector spaces • Subspaces • Intersection of subspaces • Linear spans • A sum of subspaces • The dimension of a subspace • The dimension of a sum of subspaces • The dimension of a linear span*

---

In this semester we shall transfer the results obtained in the first semester to the case of any  $n$ . In the main we shall follow the same plan of presentation as before.

Recall (see Definition 1 in Lecture 1 of [1]) that a *vector* (or *linear*) *space* over a field  $\mathbb{K}$  is a set  $\mathcal{V}$  whose members are called *vectors* and where the operation of addition  $\mathbf{x}, \mathbf{y} \mapsto \mathbf{x} + \mathbf{y}$  and the operation  $\mathbf{x} \mapsto k\mathbf{x}$  of multiplication by any number  $k \in \mathbb{K}$  are defined. It is also required that under addition  $\mathcal{V}$  should be an Abelian group and that four natural axioms should hold for multiplication by numbers in  $\mathbb{K}$ .

The concepts of a linear combination of vectors and of linearly dependent or independent families and sets of vectors have meaning in such a space. A space  $\mathcal{V}$  is said to be *finite-dimensional* if there exists in it a finite *basis*, i.e. a family of vectors in terms of which any vector of  $\mathcal{V}$  can be linearly expressed in a unique way. The number of vectors is the same in all the bases. It is called the dimension of the vector space  $\mathcal{V}$  and designated by the symbol  $\dim \mathcal{V}$ .

Let  $\mathcal{V}$  be an arbitrary finite-dimensional vector space.

**Definition 1.** A subset  $\mathcal{P}$  of a space  $\mathcal{V}$  is said to be its *subspace* if every linear combination  $k_1\mathbf{x}_1 + \dots + k_m\mathbf{x}_m$  of any vectors  $\mathbf{x}_1, \dots, \mathbf{x}_m \in \mathcal{P}$  belongs to  $\mathcal{P}$ .

It is obvious that  $\mathcal{P}$  is a subspace if and only if  $\mathbf{x} + \mathbf{y} \in \mathcal{P}$  and  $k\mathbf{x} \in \mathcal{P}$  for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{P}$  and any number  $k \in \mathbb{K}$ .

In other words, the fact that  $\mathcal{P}$  is a subspace means that the correspondences  $\mathbf{x}, \mathbf{y} \mapsto \mathbf{x} + \mathbf{y}$  and  $\mathbf{x} \mapsto k\mathbf{x}$ , where  $\mathbf{x}, \mathbf{y} \in \mathcal{P}$  and  $k \in \mathbb{K}$ , define some operations in  $\mathcal{P}$ . It is clear that under these operations *the subspace  $\mathcal{P}$  is a vector space.*  $\square$

### Examples of subspaces

1. In any vector space  $\mathcal{V}$  the one-member subset  $\{0\}$  and the whole set  $\mathcal{V}$  are subspaces. The subspace  $\{0\}$  (ordinarily denoted simply by 0) is called *zero* and the subspace  $\mathcal{V}$  is called *trivial*.

2. In the vector space  $\mathbb{K}^n$  for any  $m \leq n$  the totality of all vectors of the form  $(x^1, \dots, x^m, 0, \dots, 0)$ , whose last  $n-m$  coordinates are zero, is a subspace. This subspace is isomorphic in a natural way to the space  $\mathbb{K}^m$ .

3. In a vector space of polynomials (or, more generally, that of any functions satisfying certain conditions) a subspace is the set of all polynomials (functions) equal to zero at one or several fixed points.

4. A subspace is a set of all polynomials whose coefficients are zero for given fixed degrees, as well as a set of all even or all odd polynomials.

**Proposition 1.** *The intersection*

$$\mathcal{P} = \bigcap_{\alpha} \mathcal{P}_{\alpha}$$

*of an arbitrary family of subspaces  $\mathcal{P}_{\alpha} \subset \mathcal{V}$  is a subspace.*

*Proof.* If  $\mathbf{x}, \mathbf{y} \in \mathcal{P}$ , then  $\mathbf{x}, \mathbf{y} \in \mathcal{P}_{\alpha}$  for any  $\alpha$  and therefore  $\mathbf{x} + \mathbf{y} \in \mathcal{P}_{\alpha}$ ,  $k\mathbf{x} \in \mathcal{P}_{\alpha}$ , and hence (since  $\alpha$  is arbitrary)  $\mathbf{x} + \mathbf{y} \in \mathcal{P}$ ,  $k\mathbf{x} \in \mathcal{P}$ .  $\square$

Note that an intersection of subspaces cannot be empty since any subspace contains a zero vector 0.

If  $\mathcal{P} \cap \mathcal{Q} = 0$ , then the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  are said to be *disjoint*.

In spite of its simplicity Proposition 1 leads to important consequences.

Let  $S$  be an arbitrary subset of a vector space  $\mathcal{V}$ .

**Definition 2.** A subspace  $\mathcal{P} \subset \mathcal{V}$  is said to be *the (linear) span* of a set  $S$  if  $S \subset \mathcal{P}$  and  $\mathcal{P}$  is the smallest subspace

possessing this property, i.e. if every subspace  $\mathcal{Q}$  for which  $S \subset \mathcal{Q}$  contains  $\mathcal{P}$ . The span of a set  $S$  is designated by the symbol  $[S]$ . It is also called a subspace *generated* by the set  $S$ .

**Proposition 2.** *There exists a span  $[S]$  for any set  $S \subset \mathcal{V}$ . It is the intersection of all subspaces containing  $S$ .*

*Proof.* Since every subspace  $\mathcal{Q} \supset S$  participates in this intersection (which is a subspace, according to Proposition 1), it is contained in  $\mathcal{Q}$ . On the other hand, it obviously contains  $S$ .  $\square$

In connection with this proof the question arises: have we any right in general to speak of the intersection of subspaces containing  $S$ ? Why, strictly speaking, do such subspaces exist? The formal answer is that in accordance with the general principles of set theory the intersection of a family of subsets of an arbitrary set  $\mathcal{V}$  is well-defined even when the family is empty and is in this case, however paradoxical it may be, the whole of  $\mathcal{V}$ . But in our particular case the situation is still simpler, because the family considered is never empty. Indeed it is trivial that one of the subspaces containing  $S$  is the whole space  $\mathcal{V}$ .

A more visual description of a span  $[S]$  is given by the following proposition:

**Proposition 3.** *The span  $[S]$  of a set  $S$  consists of all possible linear combinations*

$$(1) \quad k_1 \mathbf{x}_1 + \dots + k_m \mathbf{x}_m, \quad \mathbf{x}_1, \dots, \mathbf{x}_m \in S, \quad k_1, \dots, k_m \in \mathbb{K},$$

*of the vectors of  $S$ .*

*Proof.* If  $\mathcal{P}$  is a subspace containing  $S$ , then it obviously contains all vectors of the form (1). On the other hand, it is clear that the totality of all vectors (1) is a subspace containing  $S$ .  $\square$

It follows from this proposition that *the set of vectors of a space  $\mathcal{V}$  is complete if and only if it generates the whole of  $\mathcal{V}$* .  $\square$

Recall (see Lecture 12 in [1]) that two sets of vectors are said to be *linearly equivalent* if each vector of either of the sets can be linearly expressed in terms of the vectors of the other set. It is clear that this is equivalent to saying that a vector is a linear combination of vectors of one set if and only if it is a linear combination of vectors of the other set,

i.e. according to Proposition 3, to saying that *the spans of both sets coincide* (both sets generate the same subspace).

Unlike the intersection the union of subspaces is not in general a subspace. To obtain a subspace it is necessary to pass from the union to its linear span.

**Definition 3.** A sum  $\sum_{\alpha} \mathcal{P}_{\alpha}$  of an arbitrary family of subspaces  $\mathcal{P}_{\alpha} \subset \mathcal{V}$  is the span of their union:

$$\sum_{\alpha} \mathcal{P}_{\alpha} = [\bigcup_{\alpha} \mathcal{P}_{\alpha}].$$

For two subspaces  $\mathcal{P}$  and  $\mathcal{Q}$

$$\mathcal{P} + \mathcal{Q} = [\mathcal{P} \cup \mathcal{Q}].$$

It is clear that any linear combination of the vectors of  $\mathcal{P} \cup \mathcal{Q}$  has the form  $\mathbf{x} + \mathbf{y}$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ . This proves the following proposition:

**Proposition 4.** A sum  $\mathcal{P} + \mathcal{Q}$  of the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  consists of all possible vectors of the form  $\mathbf{x} + \mathbf{y}$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ .  $\square$

A similar proposition holds also of course for the sum of any family of subspaces.

Thus far we have not used in any way the assumption about the finite dimensionality of the vector space  $\mathcal{V}$ . We shall now consider the questions where this assumption is essential.

Let  $n = \dim \mathcal{V}$ .

**Proposition 5.** For the dimension  $\dim \mathcal{P}$  of an arbitrary subspace  $\mathcal{P} \subset \mathcal{V}$  the inequality

$$\dim \mathcal{P} \leq n$$

is correct.

One may hear from students and read in some textbooks the following reasoning supposedly proving Proposition 5: any  $n + 1$  vectors of a subspace  $\mathcal{P}$ , being vectors of an  $n$ -dimensional space  $\mathcal{V}$ , are linearly dependent; therefore the subspace  $\mathcal{P}$  cannot contain more than  $n$  linearly independent vectors and so  $\dim \mathcal{P} \leq n$ .

The inadequacy of this reasoning lies in the fact that it presupposes the finite dimensionality of the space  $\mathcal{V}$ . As a matter of fact it only proves that if there is a basis in  $\mathcal{P}$  that basis contains no more than  $n$  elements. We have therefore to use another, more complicated way of reasoning to prove Proposition 5.

*Proof of Proposition 5.* If  $\mathcal{P} = 0$ , there is nothing to prove. If  $\mathcal{P} \neq 0$ , then there is a nonzero vector  $e_1 \in \mathcal{P}$ . If  $\mathcal{P} = [e_1]$ , then  $e_1$  is obviously a basis in  $\mathcal{P}$  and therefore  $\dim \mathcal{P} = 1$ . If  $\mathcal{P} \neq [e_1]$ , then there is a vector  $e_2$  in  $\mathcal{P}$  that is not linearly expressible in terms of  $e_1$ , i.e. such that the vectors  $e_1, e_2$  are linearly independent. If  $\mathcal{P} = [e_1, e_2]$ , then  $e_1, e_2$  is a basis in  $\mathcal{P}$  and hence  $\dim \mathcal{P} = 2$ . But if  $\mathcal{P} \neq [e_1, e_2]$ , then there is a vector  $e_3$  in  $\mathcal{P}$  which is not linearly expressible in terms of the vectors  $e_1, e_2$  and so on. Since  $\dim \mathcal{V} = n$ , this process must be over not later than the vector  $e_n$  appears. Consequently, the subspace  $\mathcal{P}$  is finite-dimensional and  $\dim \mathcal{P} \leq n$ .  $\square$

If  $\dim \mathcal{P} = n$ , then any basis in  $\mathcal{P}$ , being a linearly independent family consisting of  $n$  vectors, is a basis in  $\mathcal{V}$  as well. Therefore  $\mathcal{P} = \mathcal{V}$ . But if  $\dim \mathcal{P} < n$ , then the basis in  $\mathcal{P}$ , having fewer than  $n$  vectors, cannot be a complete family in  $\mathcal{V}$  and hence does not generate  $\mathcal{V}$ . Therefore  $\mathcal{P} \neq \mathcal{V}$ . Thus *a subspace  $\mathcal{P} \subset \mathcal{V}$  coincides with  $\mathcal{V}$  if and only if  $\dim \mathcal{P} = \dim \mathcal{V}$ .*  $\square$

**Theorem 1 (dimension of a sum of subspaces).** *For any two subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  the formula*

$$\dim (\mathcal{P} + \mathcal{Q}) = \dim \mathcal{P} + \dim \mathcal{Q} - \dim (\mathcal{P} \cap \mathcal{Q})$$

*is correct.*

*Proof.* Let

$$\dim \mathcal{P} = p, \quad \dim \mathcal{Q} = q, \quad \dim (\mathcal{P} \cap \mathcal{Q}) = r.$$

Consider in  $\mathcal{P} \cap \mathcal{Q}$  an arbitrary basis  $e_1, \dots, e_r$ . Adding to this basis vector after vector we finally obtain some basis

$$(2) \quad e_1, \dots, e_r, f_1, \dots, f_{p-r}$$

of the subspace  $\mathcal{P} \supset \mathcal{P} \cap \mathcal{Q}$ . Similarly in the subspace  $\mathcal{Q}$  we can construct a basis of the form

$$(3) \quad e_1, \dots, e_r, g_1, \dots, g_{q-r}.$$

Theorem 1 is obviously proved if we show that  $p + q - r$  vectors

$$(4) \quad \mathbf{e}_1, \dots, \mathbf{e}_r, \quad \mathbf{f}_1, \dots, \mathbf{f}_{p-r}, \quad \mathbf{g}_1, \dots, \mathbf{g}_{q-r}$$

form the basis of the subspace  $\mathcal{F} + \mathcal{Q}$ .

**Linear independence.** Let

$$k_1\mathbf{e}_1 + \dots + k_r\mathbf{e}_r + l_1\mathbf{f}_1 + \dots + l_{p-r}\mathbf{f}_{p-r} + \\ + m_1\mathbf{g}_1 + \dots + m_{q-r}\mathbf{g}_{q-r} = \mathbf{0}.$$

Setting

$$\begin{aligned} \mathbf{e} &= k_1\mathbf{e}_1 + \dots + k_r\mathbf{e}_r, \\ \mathbf{f} &= l_1\mathbf{f}_1 + \dots + l_{p-r}\mathbf{f}_{p-r} \\ \mathbf{g} &= m_1\mathbf{g}_1 + \dots + m_{q-r}\mathbf{g}_{q-r} \end{aligned}$$

we obtain vectors  $\mathbf{e} \in \mathcal{F} \cap \mathcal{Q}$ ,  $\mathbf{f} \in \mathcal{F}$  and  $\mathbf{g} \in \mathcal{Q}$  such that  $\mathbf{e} + \mathbf{f} + \mathbf{g} = \mathbf{0}$ . Then  $\mathbf{e} + \mathbf{f} \in \mathcal{F}$  and therefore  $\mathbf{g} = -(\mathbf{e} + \mathbf{f}) \in \mathcal{F}$ . Hence  $\mathbf{g} \in \mathcal{F} \cap \mathcal{Q}$  and consequently the vector  $\mathbf{g}$  can be linearly expressed in terms of the vectors  $\mathbf{e}_1, \dots, \mathbf{e}_r$ . But under the hypothesis the vector  $\mathbf{g}$  can be linearly expressed in terms of the vectors  $\mathbf{g}_1, \dots, \mathbf{g}_{p-r}$ . Since there can be no two distinct expressions for the same vector in terms of the basis (3) this proves that both expressions have zero coefficients. Thus  $m_1 = 0, \dots, m_{q-r} = 0$  and hence  $\mathbf{g} = \mathbf{0}$ .

But then  $\mathbf{e} + \mathbf{f} = \mathbf{0}$  and consequently (since (2) is a basis)  $k_1 = 0, \dots, k_r = 0, l_1 = 0, \dots, l_{p-r} = 0$ . This proves that the vectors (4) are linearly independent.

**Completeness.** Any vector in  $\mathcal{F} + \mathcal{Q}$  is, as we know, of the form  $\mathbf{x} + \mathbf{y}$  where  $\mathbf{x} \in \mathcal{F}$ ,  $\mathbf{y} \in \mathcal{Q}$ . On adding the expansion of the vector  $\mathbf{x}$  with respect to the basis (2) to the expansion of the vector  $\mathbf{y}$  with respect to the basis (3) we obviously obtain a representation of the vector  $\mathbf{x} + \mathbf{y}$  as a linear combination of the vectors (4). Consequently the family (4) of vectors of the subspace  $\mathcal{F} + \mathcal{Q}$  is complete.

Being linearly independent and complete, the family (4) is a basis.  $\square$

**Corollary 1.** If  $\mathcal{F} + \mathcal{Q} = \mathcal{V}$ , then  $\dim(\mathcal{F} \cap \mathcal{Q}) = p + q - n$ .

**Corollary 2.** If  $p + q > n$ , then  $\mathcal{F} \cap \mathcal{Q} \neq \mathbf{0}$ .

How can the dimension of a subspace be computed? The answer to this question depends of course on the way the subspace is given. Therefore we shall return to this question every time we come across a new way of giving subspaces. But at present we actually know one method of effective representation of subspaces, that of representing them as the linear span of a certain finite set of vectors. Therefore our general question can be stated concretely as a problem in computing the dimension  $\dim [S]$  of the span of an arbitrary (finite) set of vectors  $S$ . It is this problem that we shall now discuss.

Let  $S$  be an arbitrary finite set of vectors. We may assume without loss of generality that it contains nonzero vectors and consequently possesses linearly independent subsets. By finiteness of the number of vectors in  $S$  there are among these subsets *maximal* ones, i.e. such that joining to them any other vector of  $S$  turns them into linearly dependent sets. Since this is possible if and only if the vector to be joined is linearly expressible in terms of the vectors of the subset we deduce that *any maximal linearly independent subset  $S_0$  of the set  $S$  is linearly equivalent to the whole set  $S$ , i.e. (see above) generates the same subspace  $[S]$ . This means that a set  $S_0$  is complete in  $[S]$  and since it is, in addition, linearly independent it follows that after an arbitrary numbering it becomes a basis in  $[S]$ . So every maximal linearly independent subfamily of the set  $S$  is a basis of the span  $[S]$  of the set  $S$ .*

Since all bases of any space consist of the same number of vectors it follows in particular that all *maximal linearly independent subsets of the set  $S$  consist of the same number of vectors.*

**Definition 4.** The number of vectors of a maximal linearly independent subset of a set  $S$  is called *the rank* of the set  $S$ .

According to what has just been said this definition is correct.

In addition we see that the following proposition is true:

**Proposition 6.** *The dimension  $\dim [S]$  of the span of a set of vectors  $S$  is equal to the rank of that set.  $\square$*

On the face of it this proposition seems a vacuous tautology. In fact it has a very deep content since it identifies the number  $\dim [S]$  we are interested in with a certain number



(rank) for which there is, at least in principle, a possibility of being computed in a finite number of steps estimated in advance, i.e. such that is said to be *effectively computable*. Indeed, to compute the rank it is possible for example to look over all the subsets of the set  $S$  (there are a finite number of them!) and to determine for each subset whether it is linearly independent (which also takes a finite number of steps). Thus the significance of Proposition 6 lies in the fact that it indicates a finite procedure for computing the dimension of subspaces (when, we stress, the subspaces are given as the spans of finite—it is obligatory for effectiveness!—sets of vectors).

Of course the size of the required computation can be substantially reduced by arranging it in a reasonable way. The appropriate procedures will be dealt with in the next lecture.

# Lecture 2

*Matrix rank theorem • The rank of a matrix product • The Kronecker-Capelli theorem • Solution of systems of linear equations*

The answer to the question about the rational method for computing the rank of a set of vectors put at the end of the preceding lecture naturally depends on the way of giving these vectors. We shall consider only one but most important variant where vectors are given by their coordinates in a certain basis. This is equivalent to assuming that our vectors lie in the space of row vectors  $\mathbb{K}^n$ .

So let us be given  $m$  vectors

$$(1) \quad \begin{array}{l} \mathbf{a}_1 = (a_{11}, \dots, a_{1n}) \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \mathbf{a}_m = (a_{m1}, \dots, a_{mn}) \end{array}$$

of the space  $\mathbb{K}^n$ . Arranging the components of the vectors in the form of a rectangular matrix

$$(2) \quad A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

we can restate the problem we are interested in in the following final form:

Given a rectangular matrix (2). What is the rank of the set of its rows?

It is in this form that we shall solve the problem.

Let  $1 \leq p \leq \min(m, n)$ . On choosing in the matrix  $A$  arbitrarily  $p$  rows and  $p$  columns and considering the elements in their intersection we obtain a square "submatrix"

having  $p$  rows and  $p$  columns. The determinants of such matrices are called *the minor determinants* or *minors of order  $p$*  of the matrix  $A$ .

**Definition 1.** The highest order of nonzero minors, i.e. a number  $p$  such that there is no nonzero minor of order  $p + 1$  in the matrix  $A$  but there is a nonzero minor of order  $p$ , is called *the rank* of the matrix  $A$ .

Note that if all minors of order  $p + 1$  are zero then so are all minors of order  $p + 2$  since by the formula for the expansion of determinants any such minor is a linear combination of minors of order  $p + 1$ . Also zero of course are all minors of higher order.

It is clear that the rank  $p$  of a matrix (2) satisfies the inequalities

$$0 \leq p \leq \min(m, n),$$

with  $p = 0$  if and only if all elements of the matrix are zero.

Looking over minors of higher and higher order we can always compute the rank of an arbitrary matrix in a finite number of steps. Therefore the answer to the question put above is given by the following theorem:

**Theorem 1 (rank of a matrix).** *The rank of an arbitrary matrix is equal to the rank of the set of its rows.*

**Proof.** Note first that in any interchange of the rows of columns of a matrix  $A$  the set of all of its minors of each order is bijectively mapped onto the set of the minors of the same order of the transformed matrix, nonzero minors becoming nonzero minors. Consequently, in every such interchange the rank  $p$  of the matrix remains unchanged.

What happens to the rank of the rows? It is clear that it remains unchanged on interchange of the rows. As to interchanging the columns it reduces to a simultaneous redesignation of the components of all vectors (1), which leaves all linear relations between these vectors (or between some of them) obviously unchanged. Therefore the rank of the set of the rows of the matrix  $A$  also remains unaltered on any interchange of the columns.

Since by interchanging the rows and columns we can have a nonzero minor of order  $p$  of the matrix  $A$  occupy the top left corner it follows that in proving the equation

$p = r$  we may assume without loss of generality that

$$\Delta = \begin{vmatrix} a_{11} & \dots & a_{1p} \\ \vdots & & \vdots \\ a_{p1} & \dots & a_{pp} \end{vmatrix} \neq 0.$$

If the first  $p$  rows of the matrix  $A$  were now linearly dependent, then the rows of the determinant  $\Delta$  would obviously turn out to be linearly dependent too and so the determinant would be zero. This proves that *the rows  $a_1, \dots, a_p$  of the matrix  $A$  are linearly independent and consequently  $p \leq r$ .*

To prove the equation  $p = r$  it is therefore sufficient to establish that *any row  $\mathbf{a}_i$ , with  $i > p$ , is linearly expressible in terms of the rows  $\mathbf{a}_1, \dots, \mathbf{a}_p$ .*

To this end consider the following determinant of order  $p + 1$ :

$$(3) \quad \begin{vmatrix} a_{11} & \dots & a_{1p} & a_{1j} \\ a_{21} & \dots & a_{2p} & a_{2j} \\ \vdots & & \vdots & \vdots \\ a_{p1} & \dots & a_{pp} & a_{pj} \\ a_{i1} & \dots & a_{ip} & a_{ij} \end{vmatrix},$$

where  $1 \leq j \leq n$ . If  $1 \leq j \leq p$ , then the determinant (3) has two identical columns and is therefore zero. But if  $p + 1 \leq j \leq n$ , then the determinant (3) is the minor of the matrix  $A$  of order  $p + 1$  (resulting from the choice of the first  $p$  rows and columns besides the  $j$ th column and  $i$ th row) and therefore also zero. Consequently, on expanding this determinant by the last column we obtain for any  $j = 1, \dots, n$  an equation of the form

$$(4) \quad A_1 a_{1j} + A_2 a_{2j} + \dots + A_p a_{pj} + \Delta a_{ij} = 0,$$

where  $A_1, A_2, \dots, A_p, \Delta$  are algebraic complements of that column. These depend only on the elements in the first  $p$  columns of the determinant (3) and are in particular the same for all  $j$ . In vector notation therefore  $n$  equations (4) are equivalent to one equation of the form

$$A_1 \mathbf{a}_1 + A_2 \mathbf{a}_2 + \dots + A_p \mathbf{a}_p + \Delta \mathbf{a}_i = 0.$$

Since under the hypothesis  $\Delta \neq 0$  this proves that the vector  $\mathbf{a}_i$ ,  $p + 1 \leq i \leq n$ , is linearly expressible in terms of the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_p$ . Consequently  $r = p$ .  $\square$

The proof above shows in particular that *if the matrix  $A$  has a nonzero minor of order  $p$  possessing the property that all minors of order  $p + 1$  "bordering" it are zero, then the rank of the matrix is equal to  $p$ .*  $\square$

This remark significantly simplifies of course computing the rank.

In the particular case where the matrix  $A$  is square and its rank is equal to its order we obtain the following.

**Corollary.** *A determinant is nonzero if and only if its rows are linearly independent.*

It is clear that in transposing a matrix  $A$  the rank  $p$  remains unchanged. At the same time the rank of the rows in the transposed matrix is equal to the rank of the columns in the original matrix. This proves that *the rank of the set of rows in an arbitrary matrix is equal to the rank of the set of its columns.*  $\square$

A wonderful result relating the ranks of the families of vectors in two vector spaces having, generally speaking, even different dimensions!

What happens to the rank under matrix multiplication?

Let  $A$  be a matrix having (as above)  $n$  columns and  $m$  rows and  $B$  a matrix having  $n$  rows and  $s$  columns. Then a matrix  $AB$  is defined having  $m$  rows and  $s$  columns. If  $r(A)$  is the rank of the matrix  $A$  and  $r(B)$  is the rank of the matrix  $B$ , then what can be said about the rank  $r(AB)$  of the matrix  $AB$ ?

It turns out that in the general case one can only say that the rank  $r(AB)$  does not exceed the lower of the ranks  $r(A)$  and  $r(B)$ :

**Proposition 1.** *The inequalities*

$$r(AB) \leq r(A), \quad r(AB) \leq r(B)$$

*hold*

*Proof.* Let

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \cdot & \cdot & \cdot \\ a_{m1} & \dots & a_{mn} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & \dots & b_{1s} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{ns} \end{pmatrix},$$

$$AB = \begin{pmatrix} c_{11} & \dots & c_{1s} \\ \cdot & \cdot & \cdot \\ c_{m1} & \dots & c_{ms} \end{pmatrix}.$$

By the definition of matrix multiplication

$$c_{ik} = \sum_{j=1}^n a_{ij}b_{jk}, \quad i = 1, \dots, m, \quad k = 1, \dots, s.$$

We introduce into consideration the row vectors of matrices  $B$  and  $C$ :

$$\begin{aligned} \mathbf{b}_1 &= (b_{11}, \dots, b_{1s}), & \mathbf{c}_1 &= (c_{11}, \dots, c_{1s}), \\ &\cdot & & \cdot \\ &\cdot & & \cdot \\ \mathbf{b}_n &= (b_{n1}, \dots, b_{ns}), & \mathbf{c}_m &= (c_{m1}, \dots, c_{ms}). \end{aligned}$$

Then the formulas for  $c_{ik}$  can be rewritten in the following form:

$$\mathbf{c}_i = \sum_{j=1}^n a_{ij}\mathbf{b}_j, \quad i = 1, \dots, m$$

denoting that the vectors  $\mathbf{c}_1, \dots, \mathbf{c}_m$  are linearly expressible in terms of the vectors  $\mathbf{b}_1, \dots, \mathbf{b}_n$ . Hence

$$[\mathbf{c}_1, \dots, \mathbf{c}_m] \subset [\mathbf{b}_1, \dots, \mathbf{b}_n]$$

and therefore

$$\dim [\mathbf{c}_1, \dots, \mathbf{c}_m] \leq \dim [\mathbf{b}_1, \dots, \mathbf{b}_n],$$

i.e. by the matrix rank theorem  $r(AB) \leq r(B)$ .

The inequality  $r(AB) \leq r(A)$  can be proved in a similar way (we should only consider columns instead of rows). It is possible, however, to derive it from the inequality already proved if we take advantage of the fact that transposing leaves the rank unchanged and that  $(AB)^\top = B^\top A^\top$ . Indeed,

$$r(AB) = r((AB)^\top) = r(B^\top A^\top) \leq r(A^\top) = r(A). \quad \square$$

In the case where one of matrices  $A$  or  $B$  is square and non-singular it is possible to prove a more precise result:

**Proposition 2.** *If  $B$  is a square ( $n = s$ ) and nonsingular ( $\det B \neq 0$ ) matrix, then for any matrix  $A$*

$$r(AB) = r(A).$$

Similarly, if  $A$  is a square ( $n = m$ ) and nonsingular ( $\det A \neq 0$ ) matrix, then for any matrix  $B$

$$r(AB) = r(B).$$

In short, multiplying by a nonsingular matrix leaves the matrix rank unchanged.

*Proof.* For a nonsingular matrix  $B$  there exists an inverse matrix  $B^{-1}$  and  $A = (AB)B^{-1}$ . Therefore, according to Proposition 1,

$$r(A) = r((AB)B^{-1}) \leq r(AB).$$

Consequently  $r(A) = r(AB)$ . The equation  $r(B) = r(AB)$  for a nonsingular matrix  $A$  can be proved in a similar way.  $\square$

The theorem on the matrix rank allows us not only to compute effectively ranks and to find maximal linearly independent subsets but also helps for example to determine whether a given vector  $\mathbf{b}$  can be expressed in terms of given vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$  without having to find in explicit form the coefficients of linear dependence.

It is indeed obvious that *the vector  $\mathbf{b}$  can be linearly expressed in terms of the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$*  if and only if each maximal linearly independent subset of the set  $\mathbf{a}_1, \dots, \mathbf{a}_m$  is also a maximal linearly independent subset of the extended set  $\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}$  and hence *if the rank of the set  $\mathbf{a}_1, \dots, \mathbf{a}_m$  is equal to the rank of the set  $\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}$* .  $\square$

It is useful to restate this fact in terms of the theory of linear equations. If

$$\mathbf{a}_1 = (a_{11}, \dots, a_{1n}),$$

$$\mathbf{a}_m = (a_{m1}, \dots, a_{mn}),$$

$$\mathbf{b} = (b_1, \dots, b_n),$$

then the vector equation

$$(5) \quad x_1 \mathbf{a}_1 + \dots + x_m \mathbf{a}_m = \mathbf{b}$$

is equivalent to  $n$  numerical equations

$$(6) \quad \begin{aligned} a_{11}x_1 + \dots + a_{m1}x_m &= b_1, \\ \dots & \\ a_{1n}x_1 + \dots + a_{mn}x_m &= b_n. \end{aligned}$$

Relations (6) form a system of  $n$  nonhomogeneous linear equations in  $m$  unknowns. This system is *compatible*, i.e. has at least one solution  $x_1, \dots, x_m$  if and only if equation (5) holds, i.e. if the vector  $\mathbf{b}$  is linearly expressible in terms of the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$ .

On the other hand, by Theorem 1 the rank of the set of vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$  is equal to the rank of *the matrix of the coefficients*

$$(7) \quad \begin{pmatrix} a_{11} & \dots & a_{m1} \\ \cdot & \cdot & \cdot \\ a_{1n} & \dots & a_{mn} \end{pmatrix}$$

of system (6) and the rank of the set of vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m, \mathbf{b}$  is equal to the rank of *the augmented matrix of the coefficients*

$$(8) \quad \begin{pmatrix} a_{11} & \dots & a_{m1} & b_1 \\ \cdot & \cdot & \cdot & \cdot \\ a_{1n} & \dots & a_{mn} & b_n \end{pmatrix},$$

obtained from the matrix (7) by adding a column of free members.

This proves the following theorem.

**Theorem 2 (Kronecker-Capelli theorem).** *The system of linear equations (6) is compatible if and only if the rank of the matrix of its coefficients (7) is equal to the rank of the augmented matrix (8).*

Let system (6) be compatible. How can all of its solutions be found?

Let  $r$  be the rank of the matrix (7). On interchanging the equations and renaming (if necessary) the unknowns we may assume without loss of generality that

$$(9) \quad \Delta = \begin{vmatrix} a_{11} & \dots & a_{r1} \\ \cdot & \cdot & \cdot \\ a_{1r} & \dots & a_{rr} \end{vmatrix} \neq 0.$$

Since under the hypothesis system (6) is compatible, the rank of the matrix (8) is by the Kronecker-Capelli theorem also equal to  $r$ . This means (in view of condition (9)) that





**Stage 3.** The minor  $\Delta$  contains the coefficients of  $r$  unknowns in  $r$  equations. Leaving only these equations, assigning to the other  $n - r$  unknowns arbitrary values and obtaining in this way a system of  $r$  equations in  $r$  unknowns with a nonzero determinant we solve that system by Cramer's formulas. Thus we find the values of the other  $r$  unknowns.

The values obtained at stage 3 for the unknowns  $x_1, \dots, x_m$  are solutions of system (6) and any solution of this system can be obtained in this way.

# Lecture 3

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*Direct sums of subspaces • Decomposition of a space as a direct sum of subspaces • Factor spaces • Homomorphisms of vector spaces • Direct sums of spaces*

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Let  $\mathcal{P}$  and  $\mathcal{Q}$  be subspaces of a linear space  $\mathcal{V}$ . Recall that their sum  $\mathcal{P} + \mathcal{Q}$  consists of all vectors of the form  $\mathbf{x} + \mathbf{y}$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ .

**Definition 1.** A subspace  $\mathcal{P} + \mathcal{Q}$  is said to be a *direct sum* of the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  if each of its vectors can be uniquely represented as  $\mathbf{x} + \mathbf{y}$ ,  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ .

In this case we write  $\mathcal{P} \oplus \mathcal{Q}$  or  $\mathcal{P} \dot{+} \mathcal{Q}$  instead of  $\mathcal{P} + \mathcal{Q}$ .

**Proposition 1.** A subspace  $\mathcal{P} + \mathcal{Q}$  is a direct sum of the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  if and only if these subspaces are disjoint, i.e.  $\mathcal{P} \cap \mathcal{Q} = 0$ .

*Proof.* If we have the equation  $\mathbf{x} + \mathbf{y} = \mathbf{x}_1 + \mathbf{y}_1$ , where  $\mathbf{x}, \mathbf{x}_1 \in \mathcal{P}$  and  $\mathbf{y}, \mathbf{y}_1 \in \mathcal{Q}$ , then the vector  $\mathbf{x} - \mathbf{x}_1 = \mathbf{y}_1 - \mathbf{y}$  lies in  $\mathcal{P} \cap \mathcal{Q}$ . Therefore, if  $\mathcal{P} \cap \mathcal{Q} = 0$ , then  $\mathbf{x} = \mathbf{x}_1$  and  $\mathbf{y} = \mathbf{y}_1$ , i.e. the representation of each vector of  $\mathcal{P} + \mathcal{Q}$  as  $\mathbf{x} + \mathbf{y}$ ,  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$  is unique. Conversely, if  $\mathcal{P} \cap \mathcal{Q} \neq 0$  and  $\mathbf{a} \in \mathcal{P} \cap \mathcal{Q}$ ,  $\mathbf{a} \neq 0$ , then for any vectors  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$  we have the equation

$$\mathbf{x} + \mathbf{y} = (\mathbf{x} + \mathbf{a}) + (\mathbf{y} - \mathbf{a}),$$

where  $\mathbf{x} + \mathbf{a} \in \mathcal{P}$  and  $\mathbf{y} - \mathbf{a} \in \mathcal{Q}$ , showing that the representation of vectors of  $\mathcal{P} + \mathcal{Q}$  as  $\mathbf{x} + \mathbf{y}$ ,  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$  is not unique.  $\square$

It makes sense of course to speak of a direct sum of an arbitrary number of subspaces as well. For example, a sum  $\mathcal{P} + \mathcal{Q} + \mathcal{R}$  of three subspaces is said to be direct if the

representation of each vector of  $\mathcal{P} + \mathcal{Q} + \mathcal{R}$  as  $\mathbf{x} + \mathbf{y} + \mathbf{z}$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ ,  $\mathbf{z} \in \mathcal{R}$ , is unique. By analogy with Proposition 1 one would like to think that for this to be the case it is necessary and sufficient that the spaces  $\mathcal{P}$ ,  $\mathcal{Q}$  and  $\mathcal{R}$  should be mutually disjoint. *This is incorrect.* For any two noncollinear vectors  $\mathbf{a}$  and  $\mathbf{b}$ , for example, the subspaces  $\mathcal{P} = [\mathbf{a}]$ ,  $\mathcal{Q} = [\mathbf{b}]$ ,  $\mathcal{R} = [\mathbf{a} + \mathbf{b}]$  are mutually disjoint, nevertheless their sum  $\mathcal{P} + \mathcal{Q} + \mathcal{R} = [\mathbf{a}, \mathbf{b}]$  is not direct.

The true condition for a sum  $\mathcal{P} + \mathcal{Q} + \mathcal{R}$  to be a direct sum is given by the following proposition:

**Proposition 2.** *A sum  $\mathcal{P} + \mathcal{Q} + \mathcal{R}$  of three subspaces is their direct sum if and only if each of them is disjoint from the sum of the other two:*

$$(1) \quad \mathcal{P} \cap (\mathcal{Q} + \mathcal{R}) = 0, \quad \mathcal{Q} \cap (\mathcal{P} + \mathcal{R}) = 0, \quad \mathcal{R} \cap (\mathcal{P} + \mathcal{Q}) = 0.$$

*Proof.* If we have the equation  $\mathbf{x} + \mathbf{y} + \mathbf{z} = \mathbf{x}_1 + \mathbf{y}_1 + \mathbf{z}_1$ , where  $\mathbf{x}, \mathbf{x}_1 \in \mathcal{P}$ ,  $\mathbf{y}, \mathbf{y}_1 \in \mathcal{Q}$ ,  $\mathbf{z}, \mathbf{z}_1 \in \mathcal{R}$ , then  $\mathbf{x} - \mathbf{x}_1 = (\mathbf{y}_1 - \mathbf{y}) + (\mathbf{z}_1 - \mathbf{z}) \in \mathcal{P} \cap (\mathcal{Q} + \mathcal{R})$ . Therefore, if  $\mathbf{x}_1 \neq \mathbf{x}$ , then  $\mathcal{P} \cap (\mathcal{Q} + \mathcal{R}) \neq 0$ . Similarly, if  $\mathbf{y}_1 \neq \mathbf{y}$ , then  $\mathcal{Q} \cap (\mathcal{P} + \mathcal{R}) \neq 0$ , and if  $\mathbf{z}_1 \neq \mathbf{z}$ , then  $\mathcal{R} \cap (\mathcal{P} + \mathcal{Q}) \neq 0$ . Thus if the sum  $\mathcal{P} + \mathcal{Q} + \mathcal{R}$  is not direct, then not all conditions (1) hold. Conversely, if, for example,  $\mathcal{P} \cap (\mathcal{Q} + \mathcal{R}) \neq 0$  and  $\mathbf{a} \in \mathcal{P} \cap (\mathcal{Q} + \mathcal{R})$ ,  $\mathbf{a} \neq 0$ , then for any vectors  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ ,  $\mathbf{z} \in \mathcal{R}$  we have the equation

$$\mathbf{x} + \mathbf{y} + \mathbf{z} = (\mathbf{x} - \mathbf{a}) + (\mathbf{y} + \mathbf{b}) + (\mathbf{z} + \mathbf{c}),$$

where  $\mathbf{b} \in \mathcal{Q}$ ,  $\mathbf{c} \in \mathcal{R}$  are vectors such that  $\mathbf{a} = \mathbf{b} + \mathbf{c}$  and therefore  $\mathcal{P} + \mathcal{Q} + \mathcal{R}$  is not a direct sum.  $\square$

Of course a similar proposition is true for a sum of any number of subspaces as well.

Of particular significance is the case where  $\mathcal{P} \oplus \mathcal{Q} = \mathcal{V}$ . In this case the vector space  $\mathcal{V}$  is said to be *decomposed as a direct sum* of the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$ .

Consider the following properties of the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$ :

1° Any vector in  $\mathcal{V}$  is of the form  $\mathbf{x} + \mathbf{y}$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ , i.e.  $\mathcal{V} = \mathcal{P} + \mathcal{Q}$ .

2° The subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  are disjoint, i.e.  $\mathcal{P} \cap \mathcal{Q} = 0$ .

3° The sum of the dimensions of the subspaces  $\mathcal{F}$  and  $\mathcal{Q}$  is equal to the dimension of the space  $\mathcal{V}$ :

$$\dim \mathcal{F} + \dim \mathcal{Q} = \dim \mathcal{V}.$$

**Proposition 3.** *Any two of properties 1°, 2°, 3° imply the third.*

*Proof.* If there hold properties 1° and 2°, then by the theorem on the dimension of a sum (see Theorem 1 of Lecture 1)

$$\begin{aligned} \dim \mathcal{V} = \dim (\mathcal{F} + \mathcal{Q}) &= \dim \mathcal{F} + \dim \mathcal{Q} + \dim (\mathcal{F} \cap \mathcal{Q}) = \\ &= \dim \mathcal{F} + \dim \mathcal{Q}. \end{aligned}$$

If there hold properties 1° and 3°, then by the same theorem

$$\begin{aligned} \dim (\mathcal{F} \cap \mathcal{Q}) &= \dim (\mathcal{F} + \mathcal{Q}) - \dim \mathcal{F} - \dim \mathcal{Q} = \\ &= \dim \mathcal{V} - \dim \mathcal{F} - \dim \mathcal{Q} = 0. \end{aligned}$$

and hence  $\mathcal{F} \cap \mathcal{Q} = 0$ .

If there hold properties 2° and 3°, then again by the same theorem

$$\dim (\mathcal{F} + \mathcal{Q}) = \dim \mathcal{F} + \dim \mathcal{Q} = \dim \mathcal{V},$$

and hence  $\mathcal{F} + \mathcal{Q} = \mathcal{V}$ .  $\square$

According to Proposition 1 properties 1° and 2° imply that  $\mathcal{V} = \mathcal{F} \oplus \mathcal{Q}$ . This proves the following

**Corollary.** *The equation  $\mathcal{V} = \mathcal{F} \oplus \mathcal{Q}$  holds if and only if any two of properties 1°, 2°, 3° (and hence also the third) hold.*

**Definition 2.** If  $\mathcal{V} = \mathcal{F} \oplus \mathcal{Q}$ , then the subspaces  $\mathcal{F}$  and  $\mathcal{Q}$  are said to be *complementary*.

**Proposition 4.** *If subspaces  $\mathcal{F}$  and  $\mathcal{Q}$  are complementary, then for any basis  $\mathbf{e}_1, \dots, \mathbf{e}_p$  of the subspace  $\mathcal{F}$  and any basis  $\mathbf{e}_{p+1}, \dots, \mathbf{e}_n$  of the subspace  $\mathcal{Q}$  the vectors*

$$\mathbf{e}_1, \dots, \mathbf{e}_p, \mathbf{e}_{p+1}, \dots, \mathbf{e}_n$$

*form the basis of a space  $\mathcal{V}$ .*

*Conversely, if an arbitrary basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  is partitioned into two subfamilies  $\mathbf{e}_1, \dots, \mathbf{e}_p$  and  $\mathbf{e}_{p+1}, \dots, \mathbf{e}_n$ , then the subspaces  $\mathcal{F} = [\mathbf{e}_1, \dots, \mathbf{e}_p]$  and  $\mathcal{Q} = [\mathbf{e}_{p+1}, \dots, \mathbf{e}_n]$  are complementary.*

*Proof.* In the first statement the vectors  $e_1, \dots, e_p, e_{p+1}, \dots, e_n$  form a complete family consisting of  $n = p + q$  vectors. It is therefore a basis. In the second statement the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  have properties 1° and 3° of those indicated above. Therefore  $\mathcal{V} = \mathcal{P} \oplus \mathcal{Q}$ .  $\square$

**Corollary.** *For any subspace  $\mathcal{P} \subset \mathcal{V}$  there exists a complementary space  $\mathcal{Q}$ .*

*Proof.* Let  $e_1, \dots, e_p$  be an arbitrary basis of a subspace  $\mathcal{P}$ . Supplement this basis with some vectors  $e_{p+1}, \dots, e_n$  to form the basis of the whole space  $\mathcal{V}$ . Then a subspace  $\mathcal{Q} = [e_{p+1}, \dots, e_n]$  is complementary to  $\mathcal{P}$ .  $\square$

We see that a complementary subspace  $\mathcal{Q}$  is constructed with a lot of arbitrariness. It turns out that there exists a construction allowing us to avoid this arbitrariness (if only partially).

Let  $\mathcal{P}$  be an arbitrary subspace of a vector space  $\mathcal{V}$ .

**Definition 3.** Vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  are said to be *congruent modulo  $\mathcal{P}$*  if  $\mathbf{x} - \mathbf{y} \in \mathcal{P}$ . In this case one writes

$$\mathbf{x} \equiv \mathbf{y} \pmod{\mathcal{P}}.$$

The congruent relation is obviously an equivalence relation. Corresponding sets of vectors congruent modulo  $\mathcal{P}$  are called *cosets of the space  $\mathcal{V}$  modulo the subspace  $\mathcal{P}$* . It is clear that a set containing the vector  $\mathbf{x}$  consists of all vectors of the form  $\mathbf{x} + \mathbf{a}$ ,  $\mathbf{a} \in \mathcal{P}$ . We shall designate it by the symbol  $\mathbf{x} + \mathcal{P}$ . Another widespread designation is  $\mathbf{x} \pmod{\mathcal{P}}$ .

It is easy to see that congruences can be added together and multiplied by numbers, i.e. if

$$\mathbf{x} \equiv \mathbf{y} \pmod{\mathcal{P}} \text{ and } \mathbf{x}_1 \equiv \mathbf{y}_1 \pmod{\mathcal{P}},$$

then

$$\mathbf{x} + \mathbf{x}_1 \equiv \mathbf{y} + \mathbf{y}_1 \pmod{\mathcal{P}}$$

and

$$k\mathbf{x} \equiv k\mathbf{y} \pmod{\mathcal{P}}$$

for any number  $k \in \mathbb{K}$ . Indeed, if  $\mathbf{x} - \mathbf{y} \in \mathcal{P}$  and  $\mathbf{x}_1 - \mathbf{y}_1 \in \mathcal{P}$ , then  $(\mathbf{x} + \mathbf{x}_1) - (\mathbf{y} + \mathbf{y}_1) = (\mathbf{x} - \mathbf{y}) + (\mathbf{x}_1 - \mathbf{y}_1) \in \mathcal{P}$  and, similarly,  $k\mathbf{x} - k\mathbf{y} = k(\mathbf{x} - \mathbf{y}) \in \mathcal{P}$ .

For cosets this means that the formulas

$$(2) \quad (\mathbf{x} + \mathcal{P}) + (\mathbf{y} + \mathcal{P}) = (\mathbf{x} + \mathbf{y}) + \mathcal{P}$$

and

$$(3) \quad k(\mathbf{x} + \mathcal{P}) = k\mathbf{x} + \mathcal{P}$$

correctly define their sum and product by a number.

A direct check shows that these operations satisfy the vector space axioms. Thus under operations (2) and (3) a set of all cosets  $\mathcal{V}$  modulo  $\mathcal{P}$  is a vector space.  $\square$

**Definition 4.** This space is called *the factor space* of a space  $\mathcal{V}$  modulo a subspace  $\mathcal{P}$ . It is designated by the symbol  $\mathcal{V}/\mathcal{P}$ .

In the first semester course in algebra a similar construction was studied in detail for the case of groups and rings.

**Proposition 5.** Every subspace  $\mathcal{Q}$  complementary to a subspace  $\mathcal{P}$  is isomorphic to a factor space  $\mathcal{V}/\mathcal{P}$ .

*Proof.* Consider a mapping  $\varphi: \mathcal{Q} \rightarrow \mathcal{V}/\mathcal{P}$  defined by the formula

$$\varphi(\mathbf{x}) = \mathbf{x} + \mathcal{P}, \quad \text{where } \mathbf{x} \in \mathcal{Q}.$$

If  $\varphi(\mathbf{x}) = \varphi(\mathbf{x}_1)$ , i.e.  $\mathbf{x} + \mathcal{P} = \mathbf{x}_1 + \mathcal{P}$ , then  $\mathbf{x} - \mathbf{x}_1 \in \mathcal{P}$  and hence  $\mathbf{x} = \mathbf{x}_1$ . On the other hand, any vector  $\mathbf{z} \in \mathcal{V}$  has the form  $\mathbf{x} + \mathbf{y}$ , where  $\mathbf{x} \in \mathcal{Q}$ ,  $\mathbf{y} \in \mathcal{P}$ , and hence  $\mathbf{z} + \mathcal{P} = \mathbf{x} + \mathcal{P}$ . This proves that the mapping  $\varphi$  is bijective. Since the mapping  $\varphi$  obviously preserves sums and products by numbers, it is therefore an isomorphism.  $\square$

The geometrical fact underlying Proposition 5 is that every coset modulo  $\mathcal{P}$  has a unique vector in common with  $\mathcal{Q}$ .

Proposition 5 implies that instead of complements  $\mathcal{Q}$  we may consider the factor space  $\mathcal{V}/\mathcal{P}$  whose construction contains no arbitrariness.

It follows from Proposition 5 that

$$(4) \quad \dim \mathcal{V}/\mathcal{P} = \dim \mathcal{V} - \dim \mathcal{P}.$$

Indeed  $\dim \mathcal{V}/\mathcal{P} = \dim \mathcal{Q} = \dim \mathcal{V} - \dim \mathcal{P}$ .  $\square$

Let  $\mathcal{V}$  and  $\mathcal{W}$  be two vector spaces.

**Definition 5.** A mapping

$$\varphi: \mathcal{V} \rightarrow \mathcal{W}$$

is said to be a *linear mapping* or *homomorphism* (or simply *morphism*) of vector spaces if it preserves linear operations, i.e. if

$$\varphi(\mathbf{x} + \mathbf{y}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y})$$

and

$$\varphi(k\mathbf{x}) = k\varphi(\mathbf{x})$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  and any number  $k \in \mathbb{K}$ .

Thus the difference between homomorphisms and isomorphisms is solely in that a homomorphism is not necessarily a bijective mapping.

**Definition 6.** The totality of all vectors  $x \in \mathcal{V}$  mapping under homomorphism  $\varphi$  into the zero of the space  $\mathcal{W}$  is called *the kernel* of the homomorphism  $\varphi$  and designated by the symbol  $\text{Ker } \varphi$ . Thus

$$\text{Ker } \varphi = \{\mathbf{x} \in \mathcal{V}; \varphi(\mathbf{x}) = 0\}.$$

**Definition 7.** The totality of all vectors of  $\mathcal{W}$  having the form  $\varphi(\mathbf{x})$ ,  $\mathbf{x} \in \mathcal{V}$  is called *the image* of a homomorphism  $\varphi$  and designated by the symbol  $\text{Im } \varphi$ :

$$\text{Im } \varphi = \{\mathbf{y} \in \mathcal{W}; \mathbf{y} = \varphi(\mathbf{x})\}.$$

Sometimes  $\text{Im } \varphi$  is designated by the symbol  $\varphi(\mathcal{V})$  and called *the image of a space  $\mathcal{V}$*  under homomorphism  $\varphi$ .

It is obvious that *the sets  $\text{Ker } \varphi$  and  $\text{Im } \varphi$  are subsets* (of the sets  $\mathcal{V}$  and  $\mathcal{W}$  respectively).

The factor space  $\mathcal{W}/\text{Im } \varphi$  is designated by the symbol  $\text{Coker } \varphi$  and called *the cokernel* of a homomorphism  $\varphi$ .

A homomorphism  $\varphi$  is said to be a *monomorphism* if it is an injective mapping, i.e. if  $\varphi(\mathbf{x}) = \varphi(\mathbf{x}_1)$  when  $\mathbf{x} \neq \mathbf{x}_1$ .

A homomorphism  $\varphi$  is said to be an *epimorphism* if it maps  $\mathcal{V}$  onto  $\mathcal{W}$ , i.e. if for any vector  $\mathbf{y} \in \mathcal{W}$  there is a vector  $\mathbf{x} \in \mathcal{V}$  such that  $\mathbf{y} = \varphi(\mathbf{x})$ .

Thus a homomorphism  $\varphi$  is an isomorphism if and only if it is simultaneously a monomorphism and epimorphism.

By definition a homomorphism  $\varphi$  is an epimorphism if and only if  $\text{Im } \varphi = \mathcal{W}$ , i.e. if  $\text{Coker } \varphi = 0$ .  $\square$

Similarly it is easy to see that a homomorphism  $\varphi$  is a monomorphism if and only if  $\text{Ker } \varphi = 0$ . Indeed, if  $\varphi(\mathbf{x}) = \varphi(\mathbf{x}_1)$ , then  $\varphi(\mathbf{x} - \mathbf{x}_1) = 0$ , and therefore  $\mathbf{x} - \mathbf{x}_1 \in$



$\in \text{Ker } \varphi$ . Consequently, if  $\text{Ker } \varphi = 0$ , then  $\mathbf{x} = \mathbf{x}_1$ . Conversely, if it follows from  $\varphi(\mathbf{x}) = \varphi(\mathbf{x}_1)$  that  $\mathbf{x} = \mathbf{x}_1$ , then in particular  $\varphi(\mathbf{x}) = 0$  if and only if  $\mathbf{x} = 0$ . Consequently  $\text{Ker } \varphi = 0$ .  $\square$

If  $\text{Ker } \varphi = 0$ , then  $\varphi$  is obviously an isomorphism of a space  $\mathcal{V}$  onto a subspace  $\text{Im } \varphi \subset \mathcal{W}$ . Therefore  $\dim \text{Im } \varphi = \dim \mathcal{V}$ . It follows that if  $\text{Ker } \varphi = 0$  and  $\dim \mathcal{V} = \dim \mathcal{W}$ , then the homomorphism  $\varphi$  is an isomorphism. Indeed then  $\dim \text{Im } \varphi = \dim \mathcal{W}$ , and hence  $\text{Im } \varphi = \mathcal{W}$ .  $\square$

When  $\text{Ker } \varphi \neq 0$  it is appropriate to introduce a factor space

$$\mathcal{V}/\text{Ker } \varphi$$

which is sometimes called *the coimage* of a homomorphism  $\varphi$ . It is obvious that the formula

$$\varphi'(\mathbf{x} + \mathcal{P}) = \varphi(\mathbf{x}), \quad \mathbf{x} \in \mathcal{V}$$

correctly defines some homomorphism

$$\varphi': \mathcal{V}/\text{Ker } \varphi \rightarrow \mathcal{W}$$

called *an induced homomorphism* and it is not hard to see that the homomorphism  $\varphi'$  is an isomorphism of the factor space  $\mathcal{V}/\text{Ker } \varphi$  onto a subspace  $\text{Im } \varphi$ .

In particular we see that for any epimorphism  $\varphi: \mathcal{V} \rightarrow \mathcal{W}$  the space  $\mathcal{W}$  is isomorphic to the factor space  $\mathcal{V}/\text{Ker } \varphi$ .  $\square$

Furthermore, since  $\dim \mathcal{V}/\text{Ker } \varphi = \dim \mathcal{V} - \dim \text{Ker } \varphi$ , for any homomorphism  $\varphi: \mathcal{V} \rightarrow \mathcal{W}$  we have the formula

$$(5) \quad \dim \text{Ker } \varphi + \dim \text{Im } \varphi = \dim \mathcal{V}.$$

All these statements, except for formulas (5), are of a very general character and are correct for any groups and rings, as we know from the first semester course in algebra.

We now return to direct sums.

Let  $\mathcal{P}$  and  $\mathcal{Q}$  be arbitrary vector spaces (over the same field  $\mathbb{K}$ ). Consider the set  $\mathcal{V}$  of all pairs of the form  $(\mathbf{x}, \mathbf{y})$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ . Setting

$$(\mathbf{x}, \mathbf{y}) + (\mathbf{x}_1, \mathbf{y}_1) = (\mathbf{x} + \mathbf{x}_1, \mathbf{y} + \mathbf{y}_1)$$

and

$$k(\mathbf{x}, \mathbf{y}) = (k\mathbf{x}, k\mathbf{y}),$$

$\mathcal{V}$  obviously becomes a vector space.

**Definition 8.** The constructed space  $\mathcal{V}$  is called a *direct sum* of the spaces  $\mathcal{P}$  and  $\mathcal{Q}$  (or sometimes an *external direct sum*, to distinguish it from the “internal” direct sum considered above, when the space  $\mathcal{V}$  was preassigned and  $\mathcal{P}$  and  $\mathcal{Q}$  were its subspaces).

We are justified in using this terminology because the vectors of  $\mathcal{V}$  having the form  $(\mathbf{x}, 0)$ ,  $\mathbf{x} \in \mathcal{P}$  constitute a subspace  $\hat{\mathcal{P}}$  isomorphic to the space  $\mathcal{P}$  and those having the form  $(0, \mathbf{y})$ ,  $\mathbf{y} \in \mathcal{Q}$  constitute a subspace  $\hat{\mathcal{Q}}$  isomorphic to the space  $\mathcal{Q}$ . Besides, the subspaces  $\hat{\mathcal{P}}$  and  $\hat{\mathcal{Q}}$  are disjoint (have only the zero vector  $(0, 0)$  in common) and sum to the whole of  $\mathcal{V}$  (for  $(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, 0) + (0, \mathbf{y})$ ). Thus  $\mathcal{V} = \hat{\mathcal{P}} \oplus \hat{\mathcal{Q}}$ .

$\hat{\mathcal{P}}$  is usually identified with  $\mathcal{P}$  and  $\hat{\mathcal{Q}}$  with  $\mathcal{Q}$  and we write  $\mathcal{V} = \mathcal{P} \oplus \mathcal{Q}$  or  $\mathcal{V} = \mathcal{P} \dot{+} \mathcal{Q}$ . This causes no ambiguity.

The construction of the external direct sum was also encountered in the first semester course in algebra in connection with the case of groups. Actually, it is this construction that we used in the first semester when we constructed complexifications.

In our next lecture we consider constructions that are more specific for the theory of vector spaces.

# Lecture 4

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*The conjugate space • Dual spaces • A second conjugate space • The transformation of a conjugate basis and of the coordinates of covectors • Annulets • The space of solutions of a system of homogeneous linear equations*

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Let  $\mathcal{V}$  be an arbitrary vector space over a field  $\mathbb{K}$ .

**Definition 1.** A function  $\xi: \mathcal{V} \rightarrow \mathbb{K}$  is said to be a *linear functional* if it is a homomorphism of vector spaces i.e.

$$\xi(\mathbf{x} + \mathbf{y}) = \xi(\mathbf{x}) + \xi(\mathbf{y})$$

and

$$\xi(k\mathbf{x}) = k\xi(\mathbf{x})$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  and any number  $k \in \mathbb{K}$ . Linear functionals are also called *the covectors (covariant vectors)* of the space  $\mathcal{V}$ .

A direct check shows that a sum  $\xi + \eta$  of two linear functionals  $\xi$  and  $\eta$  (defined by the formula  $(\xi + \eta)(\mathbf{x}) = \xi(\mathbf{x}) + \eta(\mathbf{x})$ ) and a product  $k\xi$  of a linear functional  $\xi$  by an arbitrary number  $k$  (defined by the formula  $(k\xi)(\mathbf{x}) = k\xi(\mathbf{x})$ ) are linear functionals. This means that the set of all linear functionals is a subspace of the space of all functions in  $\mathcal{V}$  and hence is itself a vector space. This vector space is designated by the symbol  $\top_1(\mathcal{V})$  or  $\mathcal{V}'$ .

**Definition 2.** A vector space  $\mathcal{V}'$  is called a space *conjugate* to a space  $\mathcal{V}$ .

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be an arbitrary basis of a space  $\mathcal{V}$ .

**Proposition 1.** *The value  $\xi(\mathbf{x})$  of an arbitrary linear functional  $\xi$  on a vector  $\mathbf{x} = x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n$  is expressed by the formula*

$$(1) \quad \xi(\mathbf{x}) = \xi_1 x^1 + \dots + \xi_n x^n,$$

where

$$(2) \quad \xi_1 = \xi(\mathbf{e}_1), \dots, \xi_n = \xi(\mathbf{e}_n).$$

For any numbers  $\xi_1, \dots, \xi_n \in \mathbb{K}^n$  formula (1) uniquely gives some linear functional  $\xi \in \mathcal{V}'$  for which we have (2).

*Proof.* Formula (1) directly follows from the property of linearity

$$\begin{aligned} \xi(\mathbf{x}) &= \xi(x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n) = x^1\xi(\mathbf{e}_1) + \dots + x^n\xi(\mathbf{e}_n) = \\ &= \xi_1 x^1 + \dots + \xi_n x^n. \end{aligned}$$

Conversely, if the functional  $\xi$  is given by formula (1) then

$$\begin{aligned} \xi(\mathbf{x} + \mathbf{y}) &= \xi_1(x^1 + y^1) + \dots + \xi_n(x^n + y^n) = \\ &= \xi_1 x^1 + \dots + \xi_n x^n + \xi_1 y^1 + \dots + \xi_n y^n = \xi(\mathbf{x}) + \xi(\mathbf{y}) \end{aligned}$$

and

$$\begin{aligned} \xi(k\mathbf{x}) &= \xi_1(kx^1) + \dots + \xi_n(kx^n) = \\ &= k(\xi_1 x^1 + \dots + \xi_n x^n) = k\xi(\mathbf{x}) \end{aligned}$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  and any number  $k \in \mathbb{K}$ . Besides,

$$\xi(\mathbf{e}_i) = \xi_1 \cdot 0 + \dots + \xi_i \cdot 1 + \dots + \xi_n \cdot 0 = \xi_i. \quad \square$$

It follows from Proposition 1 that the formula

$$\mathbf{e}^i(\mathbf{e}_j) = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j, \end{cases} \quad i, j = 1, \dots, n,$$

uniquely determines  $n$  linear functionals

$$(3) \quad \mathbf{e}^1, \dots, \mathbf{e}^n.$$

It is clear that for any vector  $\mathbf{x} \in \mathcal{V}$

$$\mathbf{e}^i(\mathbf{x}) = x^i, \quad i = 1, \dots, n.$$

**Proposition 2.** *Functionals (3) form a basis of a space  $\mathcal{V}'$ . The coordinates of an arbitrary functional  $\xi$  in the basis are the coefficients (2) of its representation (1):*

$$(4) \quad \xi = \xi_1 \mathbf{e}^1 + \dots + \xi_n \mathbf{e}^n.$$

*Proof.* For any vector  $\mathbf{x} = x^1 \mathbf{e}_1 + \dots + x^n \mathbf{e}_n$  and any numbers  $\xi_1, \dots, \xi_n \in \mathbb{K}$  we have

$$\begin{aligned} (\xi_1 \mathbf{e}^1 + \dots + \xi_n \mathbf{e}^n)(\mathbf{x}) &= \xi_1 \mathbf{e}^1(\mathbf{x}) + \dots + \xi_n \mathbf{e}^n(\mathbf{x}) = \\ &= \xi_1 x^1 + \dots + \xi_n x^n. \end{aligned}$$

Consequently, if  $\xi_1, \dots, \xi_n$  are the coefficients (2) of the functional  $\xi$ , then  $(\xi_1 \mathbf{e}^1 + \dots + \xi_n \mathbf{e}^n)(\mathbf{x}) = \xi(\mathbf{x})$  for any vector  $\mathbf{x} \in \mathcal{V}$ . This proves formula (4) and the completeness of the family  $\mathbf{e}^1, \dots, \mathbf{e}^n$  in  $\mathcal{V}'$ .

On the other hand, if

$$\xi_1 \mathbf{e}^1 + \dots + \xi_n \mathbf{e}^n = 0,$$

then for any  $i = 1, \dots, n$

$$\xi_i = (\xi_1 \mathbf{e}^1 + \dots + \xi_n \mathbf{e}^n)(\mathbf{e}_i) = 0.$$

Consequently, the family  $\mathbf{e}^1, \dots, \mathbf{e}^n$  is linearly independent and hence is a basis.  $\square$

**Corollary.**

$$\dim \mathcal{V}' = \dim \mathcal{V}.$$

The basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  is said to be *conjugate* to a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .

In the Einstein notation formula (1) has the form

$$\xi(\mathbf{x}) = \xi_i x^i$$

and formula (4) has the form

$$\xi = \xi_i \mathbf{e}^i.$$

Let  $\mathcal{V}$  and  $\mathcal{W}$  be two vector spaces over a field  $\mathbb{K}$ . Suppose that any two vectors  $\mathbf{x} \in \mathcal{V}$ ,  $\mathbf{y} \in \mathcal{W}$  are assigned a number  $\langle \mathbf{x}, \mathbf{y} \rangle \in \mathbb{K}$  such that the following conditions hold:

(i) for any fixed  $y \in \mathcal{W}$  the function  $x \mapsto \langle x, y \rangle$  is a linear functional in  $\mathcal{V}$ , i.e.

$$\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle,$$

$$\langle kx, y \rangle = k \langle x, y \rangle$$

for any vectors  $x_1, x_2, x \in \mathcal{V}$  and any number  $k \in \mathbb{K}$ ;

(ii) for any fixed  $x \in \mathcal{V}$  the function  $y \mapsto \langle x, y \rangle$  is a linear functional in  $\mathcal{W}$ , i.e.

$$\langle x, y_1 + y_2 \rangle = \langle x, y_1 \rangle + \langle x, y_2 \rangle,$$

$$\langle x, ky \rangle = k \langle x, y \rangle$$

for any vectors  $y_1, y_2, y \in \mathcal{W}$  and any number  $k \in \mathbb{K}$ ;

(iii) for any vector  $x \in \mathcal{V}$  there is a vector  $y \in \mathcal{W}$  such that  $\langle x, y \rangle \neq 0$  and conversely for any vector  $y \in \mathcal{W}$  there is a vector  $x \in \mathcal{V}$  such that  $\langle x, y \rangle \neq 0$ .

Conditions (i) and (ii) are called the *bilinearity* conditions and condition (iii) is called the *nonsingularity* condition.

**Definition 3.** The function  $x, y \mapsto \langle x, y \rangle$  satisfying conditions (i), (ii), (iii) is called a *pairing* between spaces  $\mathcal{V}$  and  $\mathcal{W}$ . The spaces  $\mathcal{V}$  and  $\mathcal{W}$  for which there exists at least one pairing are said to be *dual*. The notation is  $\mathcal{V} \mid \mathcal{W}$ .

Note that the dual relation is obviously symmetrical, i.e. if  $\mathcal{V} \mid \mathcal{W}$ , then  $\mathcal{W} \mid \mathcal{V}$ .

**Proposition 3.** Any vector space  $\mathcal{V}$  is dual to a conjugate space  $\mathcal{V}'$ , i.e.

$$\mathcal{V} \mid \mathcal{V}'.$$

*Proof.* For any  $x \in \mathcal{V}$  and  $\xi \in \mathcal{V}'$  set

$$\langle x, \xi \rangle = \xi(x).$$

It is obvious that the bilinearity conditions (i) and (ii) hold (for example,  $\langle x, \xi_1 + \xi_2 \rangle = (\xi_1 + \xi_2)(x) = \xi_1(x) + \xi_2(x) = \langle x, \xi_1 \rangle + \langle x, \xi_2 \rangle$ ). The inequality  $\xi \neq 0$  implies that there exists a vector  $x \in \mathcal{V}$  such that  $\xi(x) \neq 0$ . Consequently  $\langle x, \xi \rangle \neq 0$ . Similarly the inequality  $x \neq 0$  implies that  $x^{i_0} \neq 0$  for at least one  $i_0$  and so for  $\xi = e^{i_0}$  we have  $\langle x, \xi \rangle = \xi(x) = x^{i_0} \neq 0$ . Thus condition (iii) also holds.  $\square$

The converse is true if stated as follows:

**Proposition 4.** *If spaces  $\mathcal{V}$  and  $\mathcal{W}$  are dual, then either of them is isomorphic to the space conjugate to the other:*

$$\mathcal{V} \approx \mathcal{W}', \quad \mathcal{W} \approx \mathcal{V}'.$$

*Proof.* By symmetry of the dual relation it is enough to prove only the first of these isomorphisms. Let  $\mathbf{x} \in \mathcal{V}$ . According to condition (ii) the function  $\mathbf{y} \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$  is a linear functional in  $\mathcal{W}$ , i.e. a vector of a space  $\mathcal{W}'$ . Denoting this linear functional by  $\varphi(\mathbf{x})$  we therefore obtain a certain mapping

$$\varphi: \mathcal{V} \rightarrow \mathcal{W}'.$$

Thus by definition

$$\varphi(\mathbf{x})(\mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle.$$

Therefore by condition (i)

$$\begin{aligned} \varphi(\mathbf{x}_1 + \mathbf{x}_2)(\mathbf{y}) &= \langle \mathbf{x}_1 + \mathbf{x}_2, \mathbf{y} \rangle = \\ &= \langle \mathbf{x}_1, \mathbf{y} \rangle + \langle \mathbf{x}_2, \mathbf{y} \rangle = \varphi(\mathbf{x}_1)(\mathbf{y}) + \varphi(\mathbf{x}_2)(\mathbf{y}), \end{aligned}$$

i.e.

$$\varphi(\mathbf{x}_1 + \mathbf{x}_2) = \varphi(\mathbf{x}_1) + \varphi(\mathbf{x}_2).$$

Similarly

$$\varphi(k\mathbf{x})(\mathbf{y}) = \langle k\mathbf{x}, \mathbf{y} \rangle = k\langle \mathbf{x}, \mathbf{y} \rangle = k\varphi(\mathbf{x})(\mathbf{y}),$$

i.e.

$$\varphi(k\mathbf{x}) = k\varphi(\mathbf{x}).$$

This proves that the mapping  $\varphi$  is a homomorphism.

If  $\varphi(\mathbf{x}) = 0$ , then  $\langle \mathbf{x}, \mathbf{y} \rangle = 0$  for all  $\mathbf{y} \in \mathcal{W}$  and hence (condition (iii))  $\mathbf{x} = 0$ . Thus  $\text{Ker } \varphi = 0$ . Therefore  $\text{Im } \varphi \approx \mathcal{V}$  and hence  $\dim \mathcal{V} = \dim \text{Im } \varphi \leq \dim \mathcal{W}$ .

But by symmetry of the dual relation, if the inequality  $\dim \mathcal{V} \leq \dim \mathcal{W}$  holds so must the inequality  $\dim \mathcal{W} \leq \dim \mathcal{V}$ . Consequently  $\dim \mathcal{V} = \dim \mathcal{W}$  and therefore in particular  $\dim \text{Im } \varphi = \dim \mathcal{W}$ , i.e.  $\text{Im } \varphi = \mathcal{W}$ . This proves that the homomorphism  $\varphi$  is an isomorphism.  $\square$

Since  $\mathcal{V} \mid \mathcal{V}'$  (Proposition 3) we have  $\mathcal{V}' \mid \mathcal{V}$  (symmetry), and hence  $\mathcal{V} \approx (\mathcal{V}')'$  (Proposition 4). This result is so important that deserves to be ranked as a theorem.

**Theorem 1.** *A space  $(\mathcal{V}')'$  conjugate to the conjugate is isomorphic to the original space:*

$$(\mathcal{V}')' \approx \mathcal{V}. \quad \square$$

In explicit form the isomorphism  $\mathcal{V} \rightarrow (\mathcal{V}')'$  is given by the correspondence associating with a vector  $\mathbf{x} \in \mathcal{V}$  a functional  $\hat{\mathbf{x}}$  in  $\mathcal{V}'$  defined by the formula

$$\hat{\mathbf{x}}(\xi) = \xi(\mathbf{x}).$$

As a rule the functional  $\hat{\mathbf{x}}$  is identified with the vector  $\mathbf{x}$  and therefore denoted in particular simply by  $\mathbf{x}$ .

On the face of it Theorem 1 appears to be a trivial consequence of the fact that spaces  $\mathcal{V}$  and  $(\mathcal{V}')'$  are equidimensional. But in fact it means that there is a “natural” isomorphism,  $\mathcal{V} \rightarrow (\mathcal{V}')'$ , between the spaces  $\mathcal{V}$  and  $(\mathcal{V}')'$ , that can be constructed without any arbitrariness. It is this fact that allows us to identify  $\hat{\mathbf{x}}$  and  $\mathbf{x}$  (and hence  $(\mathcal{V}')'$  and  $\mathcal{V}$ ).

Spaces  $\mathcal{V}$  and  $\mathcal{V}'$  too have the same dimension, but we cannot establish any natural isomorphism between them in the general case. At present we lack the necessary concepts for proving this statement (for example, we lack an accurate definition of what a “natural” isomorphism is) and therefore we are forced to restrict ourselves to the proof that even the simplest, one would think, and most natural attempt to construct such an isomorphism fails.

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be an arbitrary basis of a space  $\mathcal{V}$  and  $\mathbf{e}^1, \dots, \mathbf{e}^n$  a conjugate basis of a space  $\mathcal{V}'$ . We may try to consider an isomorphism  $\mathcal{V} \rightarrow \mathcal{V}'$  acting by equating the coordinates in these two bases (this isomorphism associates with every vector  $\mathbf{x} = x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n$  a covector  $\xi = x^1\mathbf{e}^1 + \dots + x^n\mathbf{e}^n$  having in the basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  the same coordinates that the vector  $\mathbf{x}$  has in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ ) hoping to find it independent of the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  (and therefore “natural”). But this hope is not realized.

To show this it is necessary to consider in a general form the transformation of the coordinates of covectors when changing the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .



We make the computations involved using the Einstein notation. To do this it is appropriate to introduce the so-called *Kronecker delta-symbol*  $\delta_i^j$  defined by the formula

$$\delta_i^j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

The main property of the symbol is expressed by the formulas

$$a^i \delta_i^j = a^j, \quad b_j \delta_i^j = b_i$$

(indeed, the terms of the left-hand sums, except, respectively, the terms  $a^j \cdot 1 = a^j$  and  $b_i \cdot 1 = b_i$ , are all zero).

With the aid of the Kronecker delta-symbol the defining property of the conjugate basis can be written as a single formula:

$$\mathbf{e}^j(\mathbf{e}_i) = \delta_i^j.$$

Similarly the fact that matrices  $(c_i^{i'})$  and  $(c_i^{i'})$  are reciprocal can be written down in the following two equivalent forms:

$$c_i^{i'} c_{i'}^j = \delta_i^j, \quad c_i^{i'} c_{j'}^{i'} = \delta_{j'}^{i'}.$$

With all this in mind consider, along with the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  another basis  $\mathbf{e}_{1'}, \dots, \mathbf{e}_{n'}$  for which

$$\mathbf{e}_{i'} = c_i^{i'} \mathbf{e}_i, \quad \mathbf{e}_i = c_i^{i'} \mathbf{e}_{i'},$$

where  $C = (c_i^{i'})$  is a transition matrix and  $C^{-1} = (c_i^{i'})$  is the inverse matrix. Then, as we know (see Lecture 11 in [1]), for the coordinates  $x^i$  and  $x^{i'}$  we have the formulas

$$x^i = c_i^{i'} x^{i'}, \quad x^{i'} = c_i^{i'} x^i.$$

Now let  $\mathbf{e}^{1'}, \dots, \mathbf{e}^{n'}$  be a basis conjugate to the basis  $\mathbf{e}_{1'}, \dots, \mathbf{e}_{n'}$ . Then by definition

$$\mathbf{e}^{j'}(\mathbf{e}_{i'}) = \delta_{i'}^{j'}.$$

Consequently

$$\mathbf{e}^{j'}(\mathbf{e}_i) = \mathbf{e}^{j'}(c_i^{i'} \mathbf{e}_{i'}) = c_i^{i'} \delta_{i'}^{j'} = c_i^{j'}.$$

But according to Proposition 2

$$\xi = \xi(e_i) e^i$$

for any covector  $\xi \in \mathcal{V}'$ . Therefore in particular

$$e^{i'} = c_i^{i'} e^i$$

and hence

$$e^i = c_i^{i'} e^{i'}$$

(the last formula can be written by symmetry or obtained by computation:  $c_i^{i'} e^{i'} = c_i^{i'} c_j^{i'} e^j = \delta_j^i e^j = e^i$ ). Similarly for the coordinates  $\xi_i = \xi(e_i)$  and  $\xi_{i'} = \xi(e_{i'})$  of an arbitrary covector  $\xi$  we have

$$\xi(e_i) = c_i^{i'} \xi(e_{i'}),$$

i.e.

$$\xi_i = c_i^{i'} \xi_{i'}$$

and, by symmetry (or by the same computation),

$$\xi_{i'} = c_{i'}^i \xi_i.$$

We see that *the covectors of a conjugate basis are transformed as the coordinates of vectors* and correspondingly *the coordinates of covectors are transformed as the vectors of a basis*.

It is customary to call the transformation of a basis *cogredient* and the transformation of the coordinates of vectors (i.e. the transformation with an inverse and a transposed matrix) *contragredient*. Thus *conjugate bases are transformed contragrediently and the coordinates of covectors are transformed cogrediently*.  $\square$

Therefore, if in some single basis (and in that conjugate to it) a vector  $x$  and covector  $\xi$  had the same coordinates, then in another basis, because the coordinates of vectors and covectors are transformed by different formulas, the vector  $x$  and covector  $\xi$  will have different coordinates. Consequently, mapping by equating the coordinates in conjugate bases is basis-dependent and so is not natural.

Let  $S \subset \mathcal{V}$  be an arbitrary subset of a vector space  $\mathcal{V}$ .

**Definition 4.** The totality of all linear functionals  $\xi \in \mathcal{V}'$  equal to zero on any vector  $\mathbf{x} \in S$  is called *an annulet of the set  $S$*  and designated by the symbol  $\text{Ann } S$  or  $S^\circ$ .

Thus

$$\text{Ann } S = \{\xi \in \mathcal{V}'; \xi(\mathbf{x}) = 0 \text{ for any } \mathbf{x} \in S\}.$$

It is obvious that  $S^\circ$  is a subspace of the space  $\mathcal{V}'$ . And if  $S \subset T$ , then  $S^\circ \supset T^\circ$ .  $\square$

**Proposition 5.** The annulet of an arbitrary set  $S \subset \mathcal{V}$  coincides with that of its linear span

$$\text{Ann } S = \text{Ann } [S].$$

*Proof.* Since  $S \subset [S]$ , then  $S^\circ \supset [S]^\circ$ . Conversely, let  $\xi \in S^\circ$ . Then for any vector  $k_1\mathbf{x}_1 + \dots + k_m\mathbf{x}_m$  of  $[S]$ , where  $\mathbf{x}_1, \dots, \mathbf{x}_m \in S$ , we have the equation

$$\xi(k_1\mathbf{x}_1 + \dots + k_m\mathbf{x}_m) = k_1\xi(\mathbf{x}_1) + \dots + k_m\xi(\mathbf{x}_m) = 0,$$

since  $\xi(\mathbf{x}_1) = 0, \dots, \xi(\mathbf{x}_m) = 0$ . Consequently,  $\xi \in [S]^\circ$ , i.e.,  $S^\circ \subset [S]^\circ$ .  $\square$

According to this proposition consideration of annulets may be restricted to subspaces.

It is clear that  $\text{Ann } 0 = \mathcal{V}'$  and conversely if  $\text{Ann } S = \mathcal{V}'$  then  $S = \{0\}$  (for if  $\xi(\mathbf{x}) = 0$  for all  $\xi \in \mathcal{V}'$ , then  $\mathbf{x} = 0$ ).  $\square$

Similarly  $\text{Ann } \mathcal{V} = 0$  and if  $\text{Ann } S = 0$ , then  $[S] = \mathcal{V}$ . Indeed, if  $[S] \neq \mathcal{V}$  and if  $\mathbf{e}_1, \dots, \mathbf{e}_n$  is a basis of the space  $\mathcal{V}$  such that  $[S] = [\mathbf{e}_1, \dots, \mathbf{e}_m]$ ,  $m < n$ , then  $\mathbf{e}^n \in [S]^\circ$  and therefore  $S^\circ \neq 0$ .  $\square$

**Proposition 6.** For any subspace  $\mathcal{F} \subset \mathcal{V}$  we have the equation

$$\dim \mathcal{F}^\circ = n - \dim \mathcal{F}.$$

*Proof.* Let  $\dim \mathcal{F} = p$  and let  $\mathbf{e}_1, \dots, \mathbf{e}_p, \dots, \mathbf{e}_n$  be a basis of the space  $\mathcal{V}$  such that  $\mathcal{F} = [\mathbf{e}_1, \dots, \mathbf{e}_p]$ . Consider a conjugate basis

$$\mathbf{e}^1, \dots, \mathbf{e}^p, \dots, \mathbf{e}^n.$$

If  $i \leq p$  and  $j > p$ , then it is automatic that  $i \neq j$  and so  $\mathbf{e}^j(\mathbf{e}_i) = 0$ . Therefore  $\mathbf{e}^{p+1}, \dots, \mathbf{e}^n \in [\mathbf{e}_1, \dots, \mathbf{e}_p]^\circ = \mathcal{F}^\circ$ . On the other hand, if  $\xi \in \mathcal{F}^\circ$ , then  $\xi(\mathbf{e}_1) = 0, \dots, \xi(\mathbf{e}_p) = 0$  and hence  $\xi = \xi_{p+1}\mathbf{e}^{p+1} + \dots + \xi_n\mathbf{e}^n$ .

This proves that the covectors  $\mathbf{e}^{p+1}, \dots, \mathbf{e}^n$  form a basis of the subspace  $\mathcal{F}^\circ$ . Therefore  $\dim \mathcal{F}^\circ = n - p$ .  $\square$

Since (Theorem 1)  $\mathcal{V} = (\mathcal{V}')'$ , in all that was said above  $\mathcal{V}$  can be replaced by  $\mathcal{V}'$  and  $\mathcal{V}'$  by  $\mathcal{V}$ . In particular, this will determine for any set  $S \subset \mathcal{V}'$  a subspace  $\text{Ann } S \subset \mathcal{V}$  consisting of vectors  $\mathbf{x} \in \mathcal{V}$  such that  $\mathbf{x}(\xi) = 0$  (i.e.  $\xi(\mathbf{x}) = 0$ ) for any covector  $\xi \in S$ , and the dimension of that subspace will be equal to  $n - r$ , where  $r$  is the dimension of the subspace  $[S]$ , i.e. the rank of the set  $S$ .

Thus, firstly, subspaces of the space  $\mathcal{V}$  can be given not only as linear spans, but also, “dually” as annulets of sets of covectors  $S = \{\xi_1, \dots, \xi_m\}$ , i.e. by equations of the form

$$(5) \quad \xi_1(\mathbf{x}) = 0, \dots, \xi_m(\mathbf{x}) = 0.$$

Secondly, we have an effective way of computing the dimension of a subspace given in this way: it is equal to  $n - r$ , where  $r$  is the rank of the set  $\{\xi_1, \dots, \xi_m\}$ .

It is appropriate to restate all this in terms of coordinates.

Covectors  $\xi_1, \dots, \xi_m$  are written in coordinates (Proposition 1) as linear forms in  $x^1, \dots, x^n$ . Therefore equations (5) take in coordinates the form

[illegible]

i.e. are ordinary homogeneous linear equations. We thus obtain the following theorem:

**Theorem 2.** *The set of all solutions  $(x^1, \dots, x^n)$  of system (6) of homogeneous linear equations is a subspace of the space  $\mathbb{K}^n$  of dimension  $n - r$  where  $r$  is the rank of a matrix of the coefficients*

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}. \quad \square$$

To find the basis of that subspace, i.e.  $n - r$  linearly independent solutions (which are usually called a *fundamental system of solutions*), it is necessary in solving system (6)

by the method described in Lecture 2 to assign to  $n - r$  "free" unknowns  $n - r$  sets of values seeing to it that linearly independent solutions result. To do this it is enough to choose the indicated sets in such a way that on being arranged as a square matrix of order  $n - r$  they should form a non-singular matrix (it is easiest to choose them in such a way that a unit matrix should result).

# Lecture 5

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*An annulet of an annulet and annulets of direct summands • Bilinear functionals and bilinear forms • Bilinear functionals in a conjugate space • Mixed bilinear functionals • Tensors*

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The fact that annulets are defined also for subsets of a space  $\mathcal{V}'$  allows us to speak of an annulet of an annulet

$$\text{Ann Ann } S = S^{\circ\circ}$$

of an arbitrary subset  $S \subset \mathcal{V}$ .

**Proposition 1.** *For any subspace  $\mathcal{P} \subset \mathcal{V}$  there holds the equation*

$$\mathcal{P}^{\circ\circ} = \mathcal{P}.$$

*Proof.* If  $\mathbf{x} \in \mathcal{P}$ , then  $\xi(\mathbf{x}) = 0$  for any  $\xi \in \mathcal{P}^\circ$ , i.e.  $\mathbf{x}(\xi) = 0$ . This means that  $\mathbf{x} \in \mathcal{P}^{\circ\circ}$ . Thus  $\mathcal{P}^{\circ\circ} \subset \mathcal{P}$  and hence  $\mathcal{P}^{\circ\circ} = \mathcal{P}$ , for  $\dim \mathcal{P}^{\circ\circ} = n - \dim \mathcal{P}^\circ = n - (n - \dim \mathcal{P}) = \dim \mathcal{P}$ .  $\square$

If, on the other hand,  $S$  is an arbitrary set, then obviously  $S^{\circ\circ} = [S]$ .

**Proposition 2.** *If  $\mathcal{V} = \mathcal{P} \oplus \mathcal{Q}$ , then  $\mathcal{V}' = \mathcal{P}^\circ \oplus \mathcal{Q}^\circ$ . And  $\mathcal{P}^\circ \approx \mathcal{Q}'$  and  $\mathcal{Q}^\circ \approx \mathcal{P}'$ .*

*Proof.* Let  $\dim \mathcal{P} = p$  and  $\dim \mathcal{Q} = q$ . Then  $p + q = n$  and  $\mathcal{P} \cap \mathcal{Q} = 0$ . Therefore  $\dim \mathcal{P}^\circ + \dim \mathcal{Q}^\circ = (n - p) + (n - q) = n$ . Besides, if  $\xi \in \mathcal{P}^\circ \cap \mathcal{Q}^\circ$ , then  $\xi(\mathbf{x}) = 0$  for any  $\mathbf{x} \in \mathcal{P}$  and  $\xi(\mathbf{y}) = 0$  for any  $\mathbf{y} \in \mathcal{Q}$ . Therefore  $\xi(\mathbf{x} + \mathbf{y}) = 0$  and hence  $\xi(\mathbf{z}) = 0$  for any  $\mathbf{z} \in \mathcal{V}$ . Consequently  $\xi = 0$ , i.e.  $\mathcal{P}^\circ \cap \mathcal{Q}^\circ = 0$ . This proves (see the corollary of Proposition 3 in Lecture 3) that  $\mathcal{V}' = \mathcal{P}^\circ \oplus \mathcal{Q}^\circ$ .

Now associate with every linear functional  $\xi \in \mathcal{P}^\circ$  its restriction

$$\xi' = \xi|_{\mathcal{Q}}$$

to the subspace  $\mathcal{Q}$ . In this way we obtain a certain mapping  $\xi \mapsto \xi'$  of the space  $\mathcal{P}^\circ$  into a space  $\mathcal{Q}'$ , which is obviously linear (a homomorphism). Its kernel consists of all functionals  $\xi \in \mathcal{P}^\circ$  for which  $\xi|_{\mathcal{Q}} = 0$ , i.e. such that  $\xi \in \mathcal{Q}^\circ$ . But according to what has been proved  $\mathcal{P}^\circ \cap \mathcal{Q}^\circ = 0$ . Therefore the mapping  $\xi \mapsto \xi'$  is a monomorphism.

Let  $\eta \in \mathcal{Q}'$ . Define in  $\mathcal{V}$  a functional  $\xi$  setting for any vector of the form  $\mathbf{x} + \mathbf{y}$ , where  $\mathbf{x} \in \mathcal{P}$ ,  $\mathbf{y} \in \mathcal{Q}$ ,

$$\xi(\mathbf{x} + \mathbf{y}) = \eta(\mathbf{y}).$$

It is clear that the functional  $\xi$  is correctly defined, linear, and belongs to  $\mathcal{P}^\circ$  and that  $\xi' = \eta$ . This proves that the mapping  $\xi \mapsto \xi'$  is an isomorphism.

The isomorphism  $\mathcal{Q}^\circ \approx \mathcal{P}'$  can be proved in a similar way.  $\square$

Note that isomorphisms of Proposition 2 are “natural”.

**Definition 1.** The function  $B: \mathbf{x}, \mathbf{y} \rightarrow B(\mathbf{x}, \mathbf{y}) \in \mathbb{K}$  of two vector arguments  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  is said to be a *bilinear functional* in  $\mathcal{V}$  if for every fixed value of one argument it is a linear functional of the other, i.e.

$$B(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = B(\mathbf{x}_1, \mathbf{y}) + B(\mathbf{x}_2, \mathbf{y}),$$

$$B(k\mathbf{x}, \mathbf{y}) = kB(\mathbf{x}, \mathbf{y})$$

and

$$B(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) = B(\mathbf{x}, \mathbf{y}_1) + B(\mathbf{x}, \mathbf{y}_2),$$

$$B(\mathbf{x}, k\mathbf{y}) = kB(\mathbf{x}, \mathbf{y})$$

for any vectors  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y} \in \mathcal{V}$  and any number  $k \in \mathbb{K}$ .

One example of a bilinear functional is a scalar product  $(\mathbf{x}, \mathbf{y})$  (see Lecture 13 in [1]). Pairings introduced in Lecture 4 are also bilinear, but their arguments are in general in different spaces. Extending the theory set forth below to this case presents no fundamental difficulties, but is rather tedious. So we shall not take it up.

Let  $e_1, \dots, e_n$  be an arbitrary basis of a space  $\mathcal{V}$ . Setting

$$(1) \quad b_{ij} = B(e_i, e_j)$$

we obtain for any two vectors  $x = x^i e_i$  and  $y = y^j e_j$  the equation

$$B(x, y) = B(e_i, e_j) x^i y^j = b_{ij} x^i y^j.$$

This proves that

$$\begin{aligned} B(x, y) &= b_{ij} x^i y^j = \\ &= \sum_{i=1}^n \sum_{j=1}^n b_{ij} x^i y^j = \\ &= b_{11} x^1 y^1 + \dots + b_{1n} x^1 y^n + \\ &+ b_{21} x^2 y^1 + \dots + b_{2n} x^2 y^n + \\ &+ b_{n1} x^n y^1 + \dots + b_{nn} x^n y^n. \end{aligned}$$

As we know (see Lecture 14 in [1]) the algebraic expression on the right is called a *bilinear form* in  $x^1, \dots, x^n$  and  $y^1, \dots, y^n$ . Thus *any bilinear functional is expressed in coordinates as a bilinear form with the coefficients (1)* (called *the coefficients of a functional B* to abridge the statements). Conversely, it is easy to see that any bilinear form gives (by formula (2)) some bilinear functional. Hence there is (for a given basis!) a bijective correspondence between bilinear functionals and bilinear forms.

The coefficients (1) of a bilinear functional  $B$  form a matrix

$$B = \begin{pmatrix} b_{11} & \dots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

which is called *the matrix of a bilinear functional B* (in a given basis).

It is clear that a sum of two bilinear functionals and a product of a bilinear functional by a number are bilinear functionals. This means that *the totality  $T_2(\mathcal{V})$  of all bilinear functionals in the space  $\mathcal{V}$  is a vector space.*  $\square$

When adding bilinear functionals their matrices are added together, and when multiplying a bilinear functional by a number its matrix is multiplied by the same number. This



means that the correspondence associating with a bilinear functional its matrix is an isomorphism of a vector space  $\mathbf{T}_2(\mathcal{V})$  onto a vector space of quadratic matrices of order  $n$ .  $\square$

With the aid of the matrix  $B$  formula (2) can be written

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top B \mathbf{y},$$

where as always

$$\mathbf{x} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} y^1 \\ \vdots \\ y^n \end{pmatrix}.$$

Cf. Lecture 14 in [1] where similar formulas were obtained for the scalar product.

Let  $\xi, \eta \in \mathbf{T}_1(\mathcal{V})$  be two linear functionals. It is clear that the formula

$$(\xi \otimes \eta)(\mathbf{x}, \mathbf{y}) = \xi(\mathbf{x}) \eta(\mathbf{y})$$

defines a certain bilinear functional  $\xi \otimes \eta$ .

**Definition 2.** A functional  $\xi \otimes \eta$  is called a *tensor product* of the functionals  $\xi$  and  $\eta$ .

Consider in particular the tensor products  $\mathbf{e}^i \otimes \mathbf{e}^j$  of covectors of a conjugate basis. Since  $\mathbf{e}^i(\mathbf{x}) = x^i$  and  $\mathbf{e}^j(\mathbf{y}) = y^j$ , we have

$$(\mathbf{e}^i \otimes \mathbf{e}^j)(\mathbf{x}, \mathbf{y}) = x^i y^j.$$

For a functional  $B' = b_{ij}(\mathbf{e}^i \otimes \mathbf{e}^j)$  we thus have the formula

$$(3) \quad B'(\mathbf{x}, \mathbf{y}) = b_{ij}(\mathbf{e}^i \otimes \mathbf{e}^j)(\mathbf{x}, \mathbf{y}) = b_{ij} x^i y^j.$$

In particular  $B'(\mathbf{e}_i, \mathbf{e}_j) = b_{ij}$ , from which it follows that the bilinear functionals  $\mathbf{e}^i \otimes \mathbf{e}^j$ ,  $i, j = 1, \dots, n$ , are linearly independent (if  $B' = 0$ , then  $b_{ij} = 0$ ). Besides, if we take an arbitrary functional  $B \in \mathbf{T}_2(\mathcal{V})$  and compose from its coefficients  $b_{ij}$  the functional  $B'$ , then according to formula (3) we have  $B' = B$ .

Thus we have proved the following proposition:

**Proposition 3.** *The tensor products*

$$\mathbf{e}^i \otimes \mathbf{e}^j, \quad i, j = 1, \dots, n,$$

of the vectors of a conjugate basis constitute a basis of a vector space  $\mathsf{T}_2(\mathcal{V})$ . The coordinates of an arbitrary bilinear functional  $B \in \mathsf{T}_2(\mathcal{V})$  in that basis are its coefficients  $b_{ij}$ :

$$B = b_{ij} \mathbf{e}^i \otimes \mathbf{e}^j. \quad \square$$

In particular we see that

$$\dim \mathsf{T}_2(\mathcal{V}) = n^2.$$

Let us now take another basis:

$$\mathbf{e}_{i'} = c_{i'}^i \mathbf{e}_i.$$

Then

$$b_{i'j'} = B(\mathbf{e}_{i'}, \mathbf{e}_{j'}) = c_{i'}^i c_{j'}^j B(\mathbf{e}_i, \mathbf{e}_j) = c_{i'}^i c_{j'}^j b_{ij}.$$

Thus, in the new basis the coefficients of a bilinear functional  $B$  are expressed by the formula

$$b_{i'j'} = c_{i'}^i c_{j'}^j b_{ij}.$$

In matrix notation this formula has the form

$$B' = C^\top B C,$$

where  $B = (b_{ij})$ ,  $B' = (b_{i'j'})$ , and  $C = (c_{i'}^i)$ . Cf. Lecture 14 in [1].

Bilinear functionals  $B: \xi, \eta \mapsto B(\xi, \eta)$  of the covectors  $\xi, \eta \in \mathcal{V}'$  are defined and studied in a quite similar way. The only change is in the position of indices. The values of every such functional are expressed by the formula

$$B(\xi, \eta) = b^{ij} \xi_i \eta_j,$$

where  $b^{ij} = B(\mathbf{e}^i, \mathbf{e}^j)$  and  $\xi_i = \xi(\mathbf{e}_i)$ ,  $\eta_j = \eta(\mathbf{e}_j)$  are the coordinates of the covectors  $\xi$  and  $\eta$ . For the other basis,  $\mathbf{e}_{i'} = c_{i'}^i \mathbf{e}_i$ , we have

$$b^{i'j'} = c_{i'}^i c_{j'}^j b^{ij}.$$

Bilinear functionals of covectors constitute a vector space

$$\mathsf{T}^2(\mathcal{V}) = \mathsf{T}_2(\mathcal{V}')$$

of dimension  $n^2$ . A tensor product  $\mathbf{x} \otimes \mathbf{y}$  of the vectors  $\mathbf{x}$  and  $\mathbf{y}$  is called a functional in  $\mathcal{T}^2(\mathcal{V})$  defined by the formula

$$(\mathbf{x} \otimes \mathbf{y})(\xi, \eta) = \xi(\mathbf{x}) \eta(\mathbf{y}).$$

Tensor products of the form

$$\mathbf{e}_i \otimes \mathbf{e}_j, \quad i, j = 1, \dots, n,$$

constitute a basis of the space  $\mathcal{T}^2(\mathcal{V})$ , with

$$B = b^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

for any functional  $B \in \mathcal{T}^2(\mathcal{V})$ .

Of greater interest is the case of bilinear functionals

$$B: \mathbf{x}, \xi \mapsto B(\mathbf{x}, \xi)$$

one argument of which is a vector  $\mathbf{x} \in \mathcal{V}$  and the other a covector  $\xi \in \mathcal{V}'$ . We shall call such functionals *mixed functionals*. They also form an  $n^2$ -dimensional vector space. We shall designate this space by the symbol  $\mathcal{T}_1^1(\mathcal{V})$ .

In coordinates the values of a mixed functional  $B$  are expressed by the formula

$$B(\mathbf{x}, \xi) = b_i^j x^i \xi_j,$$

where  $b_i^j = B(\mathbf{e}_i, \mathbf{e}^j)$ , while  $x^i = \mathbf{e}^i(\mathbf{x})$  and  $\xi_j = \xi(\mathbf{e}_j)$  are the coordinates of the vector  $\mathbf{x}$  and covector  $\xi$  (in the conjugate bases  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{e}^1, \dots, \mathbf{e}^n$ ).

On defining the tensor product  $\eta \otimes \mathbf{y}$  of the covector  $\eta$  and vector  $\mathbf{y}$  by the formula

$$(\eta \otimes \mathbf{y})(\mathbf{x}, \xi) = \eta(\mathbf{x}) \xi(\mathbf{y})$$

we immediately see that tensor products of the form  $\mathbf{e}^i \otimes \mathbf{e}_j$  constitute a basis of a space  $\mathcal{T}_1^1(\mathcal{V})$ , with

$$B = b_i^j \mathbf{e}^i \otimes \mathbf{e}_j$$

for any  $B \in \mathcal{T}_1^1(\mathcal{V})$ .  $\square$

In the basis

$$\mathbf{e}_{i'} = c_i^{i'} \mathbf{e}_i$$

the coefficients  $b_i^{j'}$  of a functional  $B \in \mathbf{T}_1^1(\mathcal{V})$  are expressed by the formula

$$(4) \quad b_i^{j'} = c_i^i c_j^{j'} b_i^j.$$

This is a type of transformation quite different from that for the coefficients of bilinear functionals of  $\mathbf{T}_2(\mathcal{V})$  or  $\mathbf{T}^2(\mathcal{V})$ . To visualize it, we shall write it in matrix notation (and at the same time derive it anew).

Let

$$B = \begin{pmatrix} b_1^1 & \dots & b_n^1 \\ \vdots & & \vdots \\ b_1^n & \dots & b_n^n \end{pmatrix}$$

and

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad \xi = (\xi_1, \dots, \xi_n).$$

Then, as can easily be seen,

$$B(x, \xi) = \xi Bx.$$

Further let

$$B' = \begin{pmatrix} b_1^{1'} & \dots & b_n^{1'} \\ \vdots & & \vdots \\ b_1^{n'} & \dots & b_n^{n'} \end{pmatrix}$$

and correspondingly

$$x' = \begin{pmatrix} x^{1'} \\ \vdots \\ x^{n'} \end{pmatrix}, \quad \xi' = (\xi_1', \dots, \xi_{n'}'),$$

and so

$$B(x, \xi) = \xi' B' x'.$$

As we know,

$$x = Cx'$$

and

$$\xi' = \xi C, \quad \text{i.e.} \quad \xi = \xi' C^{-1}$$

(the coordinates of covectors are transformed cogrediently). Therefore

$$\xi Bx = \xi' C^{-1} B C x' = \xi' B' x',$$

i.e.

$$B' = C^{-1} B C.$$

This is precisely formula (4) in matrix notation. Instead of the transposed matrix  $C^T$  there has appeared the inverse matrix  $C^{-1}$ .

A generalization suggests itself.

**Definition 3.** A  $(p, q)$ -tensor in a space  $\mathcal{V}$ , where  $p, q \geq 0$ , is an arbitrary function

$$T : \mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q \mapsto T(\mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q)$$

of  $p$  vector arguments  $\mathbf{x}_1, \dots, \mathbf{x}_p$  and  $q$  covector arguments  $\xi^1, \dots, \xi^q$ , which is linear in every argument (with the values of the others fixed), that is to say, *multilinear*.

Thus bilinear functionals of vectors are  $(2, 0)$ -tensors, bilinear functionals of covectors are  $(0, 2)$ -tensors, and mixed bilinear functionals are  $(1, 1)$ -tensors.

Similarly, covectors are  $(1, 0)$ -tensors, and vectors, by virtue of the identification  $\mathcal{V} = (\mathcal{V}')'$  are  $(0, 1)$ -tensors.

According to the general conventions about functions  $(0, 0)$ -tensors having no arguments at all are identified with elements of the field  $\mathbb{K}$ .

The set of all  $(p, q)$ -tensors is designated by the symbol  $\mathbf{T}_p^q(\mathcal{V})$ , a zero index being dropped. This is in agreement with the notation  $\mathbf{T}_2(\mathcal{V})$ ,  $\mathbf{T}^2(\mathcal{V})$ , and  $\mathbf{T}_1^1(\mathcal{V})$  introduced above for spaces of bilinear functionals, as well as with the notation  $\mathbf{T}_1(\mathcal{V})$  introduced for a conjugate space  $\mathcal{V}'$ . According to what has been said above  $\mathbf{T}^1(\mathcal{V}) = \mathcal{V}$  and  $\mathbf{T}_0^0(\mathcal{V}) = \mathbb{K}$ .

It is clear that each of the sets  $\mathbf{T}_p^q(\mathcal{V})$  is a vector space (under the ordinary linear operations on functions).

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be an arbitrary basis of a space  $\mathcal{V}$  and  $\mathbf{e}^1, \dots, \mathbf{e}^n$  a conjugate basis of a space  $\mathcal{V}'$ . Also let

$$\begin{aligned} \mathbf{x}_1 &= x_1^{i_1} \mathbf{e}_{i_1}, \dots, \mathbf{x}_p = x_p^{i_p} \mathbf{e}_{i_p} \\ \xi^1 &= \xi_{j_1}^1 \mathbf{e}^{j_1}, \dots, \xi^q = \xi_{j_q}^q \mathbf{e}^{j_q} \end{aligned}$$

Then by multilinearity

$$(5) \quad T(\mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q) = \sum_{i_1, \dots, i_p, j_1, \dots, j_q} T_{i_1 \dots i_p}^{j_1 \dots j_q} x_1^{i_1} \dots x_p^{i_p} \xi_1^{j_1} \dots \xi_q^{j_q},$$

where

$$(6) \quad T_{i_1 \dots i_p}^{j_1 \dots j_q} = T(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_p}, \mathbf{e}^{j_1}, \dots, \mathbf{e}^{j_q}),$$

The numbers  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$  are called *the coefficients* of a tensor  $T$ . Their number is equal to  $n^{p+q}$ .

To reduce the formulas it is convenient to introduce *the composite indices*

$$\alpha = (i_1, \dots, i_p) \quad \text{and} \quad \beta = (j_1, \dots, j_q).$$

Setting

$$T_\alpha^\beta = T_{i_1, \dots, i_p}^{j_1, \dots, j_q}$$

and

$$x^\alpha = x_1^{i_1} \dots x_p^{i_p}, \quad \xi_\beta = \xi_{j_1}^1 \dots \xi_{j_q}^q,$$

we can write formula (5) in the following reduced form:

$$(7) \quad T(\mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q) = T_\alpha^\beta x^\alpha \xi_\beta.$$

This formula means that in coordinates any tensor can be expressed as a *multilinear form*  $T_\alpha^\beta x^\alpha \xi_\beta$ .

Conversely, every multilinear form  $T_\alpha^\beta x^\alpha \xi_\beta = T_{i_1 \dots i_p}^{j_1 \dots j_q} x_1^{i_1} \dots x_p^{i_p} \xi_1^{j_1} \dots \xi_q^{j_q}$  gives by formula (7) some functional

$$T : \mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q \mapsto T(\mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q)$$

which is obviously multilinear, i.e. a tensor.

Thus, for a given basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  of a space  $\mathcal{V}$   $(p, q)$ -tensors are in bijective correspondence to multilinear forms  $T_\alpha^\beta x^\alpha \xi_\beta$ , i.e. sets  $(T_\alpha^\beta) = (T_{i_1 \dots i_p}^{j_1 \dots j_q})$  of elements of the field  $\mathbb{K}$ .  $\square$

Let us now transform from the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  to a new basis  $\mathbf{e}_1', \dots, \mathbf{e}_n'$ . Let

$$\mathbf{e}_i' = c_i^j \mathbf{e}_j.$$

Then, by multilinearity, for the coefficients

$$T_{i'_1 \dots i'_p}^{j'_1 \dots j'_q} = T(e_{i'_1}, \dots, e_{i'_p}, e^{j'_1}, \dots, e^{j'_q})$$

of the tensor  $T$  in the basis  $e_1, \dots, e_n$ , we have

$$(8) \quad T_{i'_1 \dots i'_p}^{j'_1 \dots j'_q} = c_{i'_1}^{i_1} \dots c_{i'_p}^{i_p} c_{j_1}^{j'_1} \dots c_{j_q}^{j'_q} T_{i_1 \dots i_p}^{j_1 \dots j_q}$$

This is the so-called *tensor transformation law*. We may say by convention that in formula (8) each index is transformed irrespective of the others, with the subscripts transformed cogrediently and the superscripts transformed contragrediently.

In contracted notation formula (8) has the form

$$T_{\alpha'}^{\beta'} = c_{\alpha'}^{\alpha} c_{\beta}^{\beta'} T_{\alpha}^{\beta},$$

where

$$c_{\alpha'}^{\alpha} = c_{i'_1}^{i_1} \dots c_{i'_p}^{i_p}, \quad c_{\beta}^{\beta'} = c_{j_1}^{j'_1} \dots c_{j_q}^{j'_q}.$$

**Theorem 1.** Suppose every basis  $e_1, \dots, e_n$  of a space  $\mathcal{V}$  has associated with it  $n^{p+q}$  numbers  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$ , numbers associated with different bases being related to one another by the tensor transformation law (8). Then there exists in the space  $\mathcal{V}$  a unique  $(p, q)$ -tensor the coefficients of which in each basis  $e_1, \dots, e_n$  are the given numbers  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$ .

*Proof.* As was already stated above, the giving of numbers  $T_{i_1 \dots i_p}^{j_1 \dots j_q} = T_{\alpha}^{\beta}$  in a given basis  $e_1, \dots, e_n$  determines by the formula

$$\begin{aligned} T(x_1, \dots, x_p, \xi^1, \dots, \xi^q) &= \\ &= T_{i_1 \dots i_p}^{j_1 \dots j_q} x_1^{i_1} \dots x_p^{i_p} \xi_1^{j_1} \dots \xi_q^{j_q} = T_{\alpha}^{\beta} x^{\alpha} \xi_{\beta} \end{aligned}$$

some  $(p, q)$ -tensor  $T$ . To prove Theorem 1 it is therefore sufficient to verify that in any other basis  $e_1', \dots, e_n'$  that tensor has the given coefficients. But this is obvious, for

according to that which has been proved above the coefficients of the tensor  $T$  in the basis  $\mathbf{e}_1', \dots, \mathbf{e}_n'$  are the numbers  $c_\alpha^\alpha c_\beta^{\beta'} T_\alpha^\beta$ , and under the hypothesis these numbers are precisely equal to  $T_{\alpha'}^{\beta'}$ .  $\square$

According to this theorem tensors can be identified with sets of numbers  $(T_{i_1 \dots i_p}^{j_1 \dots j_q})$  related by formulas (6). It is in this form that tensors usually appear in physics. In this interpretation numbers  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$  are generally termed not coefficients of tensors, but *tensor components*.



# Lecture 6

---

*Multiplication of tensors • The basis of a space of tensors • Contraction of tensors • The rank space of a multilinear functional*

---

As was already noted in the preceding lecture, tensors of the same type can be added together. It is clear that in doing so their coefficients (components) are also added:

$$(T + S)_{i_1 \dots i_p}^{j_1 \dots j_q} = T_{i_1 \dots i_p}^{j_1 \dots j_q} + S_{i_1 \dots i_p}^{j_1 \dots j_q}.$$

When tensors are interpreted as sets of numbers  $T_{i_1 \dots i_p}^{j_1 \dots j_q}$  this formula is taken as the definition of their sum.

However, defined for tensors besides the operation of addition is also *the operation of multiplication* which is designated by the symbol  $\otimes$ . We can multiply any  $(p, q)$ - and  $(r, s)$ -tensors to obtain as a result a  $(p + r, q + s)$ -tensor. On components multiplication is defined by the formula

$$(T \otimes S)_{i_1 \dots i_{p+r}}^{j_1 \dots j_{q+s}} = T_{i_1 \dots i_p}^{j_1 \dots j_q} S_{i_{p+1} \dots i_{p+r}}^{j_{q+1} \dots j_{q+s}}$$

(each component of a tensor  $T$  is thus multiplied by a tensor  $S$ ) or, when tensors are interpreted as multilinear functions, by the formula

$$\begin{aligned} (T \otimes S)(\mathbf{x}_1, \dots, \mathbf{x}_{p+r}, \xi^1, \dots, \xi^{q+s}) &= \\ &= T(\mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q) S(\mathbf{x}_{p+1}, \dots, \mathbf{x}_{p+r}, \xi^{q+1}, \dots, \xi^{q+s}). \end{aligned}$$

It is obvious that  $\otimes$ -multiplication is distributive over addition

$$\begin{aligned} (T + S) \otimes R &= T \otimes R + S \otimes R, \\ R \otimes (T + S) &= R \otimes T + R \otimes S, \end{aligned}$$

and associative:

$$(T \otimes S) \otimes R = T \otimes (S \otimes R).$$

But in general it is noncommutative:

$$T \otimes S \neq S \otimes T.$$

If one (or both) of the cofactors is a  $(0, 0)$ -tensor, i.e. a number  $k$ , then the tensor product coincides with the ordinary one:

$$k \otimes T = T \otimes k = kT.$$

Under  $+$  and  $\otimes$  operations all vector spaces  $T_p^q(\mathcal{V})$  constitute an algebraic object that is an example of the so-called *twice graded algebra*. This algebra is designated by the symbol  $T(\mathcal{V})$  and called *the tensor algebra* of a vector space  $\mathcal{V}$ .

Let as always  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be an arbitrary basis of a space  $\mathcal{V}$  and  $\mathbf{e}^1, \dots, \mathbf{e}^n$  a conjugate basis of a space  $\mathcal{V}'$ .

For any composite indices  $\alpha = (i_1, \dots, i_p)$  and  $\beta = (j_1, \dots, j_q)$  we set

$$\mathbf{e}^\alpha = \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p}, \quad \mathbf{e}_\beta = \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_q}.$$

Then

$$\mathbf{e}^\alpha(\mathbf{x}_1, \dots, \mathbf{x}_p) = x_1^{i_1} \dots x_p^{i_p} = x^\alpha$$

and similarly

$$\mathbf{e}_\beta(\xi^1, \dots, \xi^q) = \xi_{j_1}^1 \dots \xi_{j_q}^q = \xi_\beta$$

for any vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  and any covectors  $\xi^1, \dots, \xi^q$ . Therefore for any numbers  $T_\alpha^\beta$  we have

$$(T_\alpha^\beta \mathbf{e}^\alpha \otimes \mathbf{e}_\beta)(\mathbf{x}_1, \dots, \mathbf{x}_p, \xi^1, \dots, \xi^q) = T_\alpha^\beta x^\alpha \xi_\beta.$$

This proves the following proposition.

**Proposition 1.** *All possible tensor products of the form*

$$\mathbf{e}^\alpha \otimes \mathbf{e}_\beta = \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p} \otimes \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_q}$$

constitute a basis of a space  $\mathcal{T}_p^q(\mathcal{V})$ . The coordinates of the tensor  $T$  in this basis are its coefficients:

$$\begin{aligned} T &= T_{i_1 \dots i_p}^{j_1 \dots j_q} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p} \otimes \mathbf{e}_{j_1} \otimes \dots \otimes \mathbf{e}_{j_q} = \\ &= T_{\alpha}^{\beta} \mathbf{e}^{\alpha} \otimes \mathbf{e}_{\beta}. \quad \square \end{aligned}$$

For the case of bilinear functionals we already know this proposition from the preceding lecture.

In particular we see that

$$\dim \mathcal{T}_p^q(\mathcal{V}) = n^{p+q},$$

so that the dimension of a space of  $(p, q)$ -tensors equals, as was to be expected, the number of their components.

Let  $T$  be an arbitrary  $(p, q)$ -tensor, where  $p > 0$  and  $q > 0$ , and let  $1 \leq k \leq p$ ,  $1 \leq l \leq q$ . On substituting in the tensor  $T$  a vector of a basis  $\mathbf{e}_i$  for the  $k$ th vector argument and a covector  $\mathbf{e}^i$  for the  $l$ th covector argument and carrying out summation over  $i$  (from 1 to  $n$ ) we obtain one new  $(p-1, q-1)$ -tensor. Thus

$$\begin{aligned} S(\mathbf{x}_1, \dots, \mathbf{x}_{p-1}, \xi^1, \dots, \xi^{q-1}) &= \\ &= T(\mathbf{x}_1, \dots, \mathbf{x}_{k-1}, \mathbf{e}_i, \mathbf{x}_k, \dots, \mathbf{x}_{p-1}, \xi^1, \dots, \\ &\quad \dots, \xi^{l-1}, \mathbf{e}^i, \xi^l, \dots, \xi^{q-1}), \end{aligned}$$

the right-hand side implying according to the Einstein convention summation over  $i$ . The components of the tensor  $S$  are obviously expressible by the formula

$$S_{i_1 \dots i_{p-1}}^{j_1 \dots j_{q-1}} = T_{i_1 \dots i_{k-1} i i_{k \dots i_{p-1}}}^{j_1 \dots j_{l-1} i j_l \dots j_{q-1}}.$$

**Definition 1.** The constructed tensor  $S$  is called a *contraction of a tensor  $T$  over the  $k$ th subscript and the  $l$ th superscript*.

It is necessary to verify that this definition is correct, i.e. that a tensor  $S$  is independent of the choice of a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . But this is easily done. Indeed, if  $\mathbf{e}_1', \dots, \mathbf{e}_n'$  is any other basis and

$$\mathbf{e}_i' = c_i^j \mathbf{e}_j,$$

then on replacing “noncontractible” arguments by dots, dots, dots we get

$$\begin{aligned}
 T(x_1, \dots, x_{k-1}, e_{i'}, x_k, \dots, x_{p-1}, \xi^1, \dots \\
 \dots, \xi^{l-1}, e^{i'}, \xi^l, \dots, \xi^{q-1}) = \\
 = T(\dots, e_{i'}, \dots, e^{i'}, \dots) = \\
 = c_i^i c_j^{j'} T(\dots, e_i, \dots, e^j, \dots) = \\
 = \delta_j^i T(\dots, e_i, \dots, e^j, \dots) = \\
 = T(\dots, e_i, \dots, e^j, \dots) = \\
 = S(x_1, \dots, x_{p-1}, \xi^1, \dots, \xi^{q-1}). \quad \square
 \end{aligned}$$

### Examples of contractions.

1. On contracting a mixed bilinear functional  $B(x, \xi) = b_i^j x^i \xi_j$  over the only subscript and the only superscript we obtain a  $(0, 0)$ -tensor, i.e. a number  $B(e_i, e^i)$ . This number is called the trace of the functional and designated by the symbol  $\text{tr } B$ . Thus by definition

$$\text{tr } B = b_i^i = b_1^1 + \dots + b_n^n,$$

whence we see that *the trace of a functional is equal to the trace of its matrix*, i.e. to the sum of the diagonal elements of the matrix.

2. In particular, for any vector  $x$  and covector  $\xi$

$$\text{tr}(\xi \otimes x) = \xi_i x^i = \xi_1 x^1 + \dots + \xi_n x^n.$$

3. Let  $T$  be an arbitrary  $(p, q)$ -tensor. By taking  $p$  vectors  $x_1, \dots, x_p$  and  $q$  covectors  $\xi^1, \dots, \xi^q$  we can construct a  $(p + q, p + q)$ -tensor

$$x_1 \otimes \dots \otimes x_p \otimes T \otimes \xi^1 \otimes \dots \otimes \xi^q.$$

On contracting the tensor  $p + q$  times over the subscripts and superscripts with the same numbers we obviously obtain a number

$$T_{i_1 \dots i_p}^{j_1 \dots j_q} x_1^{i_1} \dots x_p^{i_p} \xi_{j_1}^1 \dots \xi_{j_q}^q.$$

i.e. the value of the tensor  $T$  on the vectors  $x_1, \dots, x_p$  and covectors  $\xi^1, \dots, \xi^q$ .

Of particular interest are  $(p, 0)$ -tensors also called *multilinear functionals* in a vector space  $\mathcal{V}$ . The number  $p$  of arguments is called *the degree* of a functional.

For a chosen basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  every multilinear functional  $A$  of degree  $p$  is uniquely determined by its coefficients

$$A_{i_1 \dots i_p} = A(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_p})$$

using either of the two equivalent formulas:

$$A(\mathbf{x}_1, \dots, \mathbf{x}_p) = A_{i_1 \dots i_p} x_1^{i_1} \dots x_p^{i_p}$$

or

$$A = A_{i_1 \dots i_p} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p}.$$

If we fix in a functional  $A$  all arguments but one, the result is a functional of degree 1, i.e. a covector.

**Definition 2.** Every such covector is called a covector *associated* with a multilinear functional  $A$ .

To obtain an arbitrary associated covector  $\xi$  it is necessary to give  $p - 1$  vectors  $\mathbf{a}_1, \dots, \mathbf{a}_{p-1}$  and a number  $i$ . The covector  $\xi$  is then given by the formula

$$\xi(\mathbf{x}) = A(\mathbf{a}_1, \dots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_i, \dots, \mathbf{a}_{p-1}).$$

**Definition 3.** The subspace of a space  $\mathcal{V}'$  generated by all covectors associated with a multilinear functional  $A$  is called *the rank space* of that functional.

**Definition 4.** A multilinear functional  $A$  of degree  $p$  is said to be *expressible in tensor form* in terms of covectors  $\xi^1, \dots, \xi^r$  if it is a linear combination of tensor products of the form  $\xi^{j_1} \otimes \dots \otimes \xi^{j_p}$ , where  $1 \leq j_1, \dots, j_p \leq r$ .

**Proposition 2.** Any multilinear functional  $A$  is expressible in tensor form in terms of every basis of its rank space.

*Proof.* Let  $\mathcal{R}$  be the rank space of a functional  $A$  and  $\mathbf{e}^1, \dots, \mathbf{e}^r$  be its arbitrary basis. Supplement this basis to obtain a basis

$$\mathbf{e}^1, \dots, \mathbf{e}^r, \dots, \mathbf{e}^n$$

of the whole space  $\mathcal{V}'$  and consider a conjugate basis

$$\mathbf{e}_1, \dots, \mathbf{e}_r, \dots, \mathbf{e}_n$$

of a space  $\mathcal{V}' = (\mathcal{V}')'$ . As we know (see Lecture 4), the vectors

$$\mathbf{e}_{r+1}, \dots, \mathbf{e}_n$$

constitute a basis of the annulet  $\mathcal{H}^0$  of a subspace  $\mathcal{H}$  and hence

$$\xi(\mathbf{e}_j) = 0 \quad \text{when } j > r$$

for any covector  $\xi \in \mathcal{R}$ . But contained among the covectors of  $\mathcal{R}$  are in particular all covectors of the form

$$\xi(\mathbf{x}) = A(\mathbf{x}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_p}).$$

For any indices  $i_2, \dots, i_p$  and any index  $i_1 > r$  we have therefore

$$A(\mathbf{e}_{i_1}, \mathbf{e}_{i_2}, \dots, \mathbf{e}_{i_p}) = 0,$$

i.e.

$$A_{i_1 i_2 \dots i_p} = 0.$$

It can be proved in a similar way that this last equation holds not only for  $i_1 > r$ , but also for  $i_2 > r$  and in general whenever  $i_k > r$  at least for one  $k = 1, 2, \dots, p$ . But then we may consider that in the expansion

$$A = A_{i_1 \dots i_p} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p}$$

summation over all the indices takes place only from 1 to  $r$ , and this precisely means that in tensor form a functional  $A$  is expressible in terms of a basis  $\mathbf{e}^1, \dots, \mathbf{e}^r$ .  $\square$

# Lecture 7

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*The rank of a multilinear functional · Functionals and permutations · Alternation*

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Let us continue the study of the rank space of a multilinear functional.

Let  $A$  be a multilinear functional and  $\mathcal{R}$  its rank space. Further let  $\xi^1, \dots, \xi^r$  be an arbitrary family of covectors in terms of which the functional  $A$  is expressible in tensor form.

**Proposition 1.** The subspace  $\mathcal{R}$  is contained in the linear span of covectors  $\xi^1, \dots, \xi^r$ :

$$\mathcal{R} \subset [\xi^1, \dots, \xi^r].$$

*Proof.* Under the hypothesis we have

$$A = b_{i_1 \dots i_p} \xi^{i_1} \otimes \dots \otimes \xi^{i_p},$$

where  $b_{i_1 \dots i_p}$  are some numbers, and summation over  $i_1, \dots, i_p$  takes place from 1 to  $r$ .

An arbitrary covector

$$\xi(\mathbf{x}) = A(\mathbf{a}_1, \dots, \mathbf{a}_{s-1}, \mathbf{x}, \mathbf{a}_s, \dots, \mathbf{a}_{p-1})$$

associated with the functional  $A$  can therefore be expressed by the formula

$$\xi = c_q \xi^q,$$

where

$$c_q = b_{i_1 \dots i_{s-1} q i_s \dots i_{p-1}} a_1^{i_1} \dots a_{s-1}^{i_{s-1}} a_s^{i_s} \dots a_{p-1}^{i_{p-1}}.$$

Consequently  $\xi \in [\xi^1, \dots, \xi^q]$  and hence  $\mathcal{R} \subset [\xi^1, \dots, \xi^q]$ .  $\square$

**Definition 1.** The dimension of the rank space  $\mathcal{R}$  is called *the rank* of a multilinear functional  $A$ .

**Theorem 1.** *The rank of a multilinear functional  $A$  is equal to the smallest number of covectors in terms of which the functional  $A$  can be expressed in tensor form, i.e.*

(a) *if the functional  $A$  is expressible in tensor form in terms of covectors  $\xi^1, \dots, \xi^r$ , then its rank does not exceed  $r$ ;*

(b) *if  $r$  is the rank of the functional  $A$ , then there exist  $r$  covectors  $\xi^1, \dots, \xi^r$  in terms of which the functional  $A$  is expressible in tensor form.*

*Moreover, the family of covectors  $\xi^1, \dots, \xi^r$  possesses the property indicated in (b) if and only if it is a basis of the rank space  $\mathcal{R}$ .*

*Proof.* According to Proposition 2 of the preceding lecture, in tensor form the functional  $A$  can be expressed in terms of a basis of the space  $\mathcal{R}$ . This proves, in particular, statement (b).

If, on the other hand, the functional  $A$  can be expressed in tensor form in terms of covectors  $\xi^1, \dots, \xi^r$  and therefore, according to Proposition 1, we have the inclusion

$$\mathcal{R} \subset [\xi^1, \dots, \xi^r],$$

then

$$\dim \mathcal{R} \leq \dim [\xi^1, \dots, \xi^r] \leq r.$$

This proves statement (a).

In addition we see that when  $r = \dim \mathcal{R}$  there must necessarily hold the equation

$$\mathcal{R} = [\xi^1, \dots, \xi^r]$$

showing that the family of covectors  $\xi^1, \dots, \xi^r$  (obviously, linearly independent) is a basis of the space  $\mathcal{R}$ .

This completes the proof of all the statements of Theorem 1.  $\square$

It goes without saying that all this remains valid (with obvious modifications) for functionals of covectors  $((0, p)$ -tensors). One should only keep in mind that it is *vectors* that are associated with such functionals, so that the rank space turns out to be a subspace of a vector space  $\mathcal{V}$ .



Recall that a *permutation of degree  $p$*  is an arbitrary bijective mapping of a set  $\{1, \dots, p\}$  onto itself. Any such permutation  $\sigma$  is usually represented by a two-row array

$$\begin{pmatrix} 1 & 2 & \dots & p \\ \sigma(1) & \sigma(2) & \dots & \sigma(p) \end{pmatrix},$$

although in general the lower row alone would be quite enough.

All permutations of degree  $p$  form a group (under composition) which is called a *symmetric group* and designated by the symbol  $S_p$ .

Permutations are divided into even and odd ones according to the number of pairs  $(\sigma(i), \sigma(j))$  for which  $i < j$  but  $\sigma(i) > \sigma(j)$  is even or odd.

The *sign of permutation* is the number  $+1$  if the permutation is even and the number  $-1$  if the permutation is odd. We shall designate the sign of a permutation  $\sigma$  by the symbol  $\varepsilon_\sigma$ .

It is known that

$$\varepsilon_{\sigma\tau} = \varepsilon_\sigma \varepsilon_\tau$$

for any two permutations  $\sigma$  and  $\tau$ , from which it follows in particular that *all even permutations constitute a subgroup of the group  $S_p$* .

Let  $A$  be a multilinear functional of degree  $p$ .

**Definition 2.** For any permutation  $\sigma \in S_p$  the symbol  $\sigma A$  stands for a functional given by the formula

$$(\sigma A)(\mathbf{x}_1, \dots, \mathbf{x}_p) = A(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(p)}).$$

It is clear that

$$(\sigma A)_{i_1 \dots i_p} = A_{i_{\sigma(1)} \dots i_{\sigma(p)}}.$$

*In order to obtain the coefficients of a functional  $\sigma A$  it is thus necessary to apply a permutation  $\sigma$  to the indices of the functional  $A$ .*

**Example.** If  $n = 5$ ,  $p = 3$  and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix},$$

then

$$(\sigma A)_{145} = A_{514}, \quad (\sigma A)_{553} = A_{355}.$$

It is obvious that for any permutation  $\sigma$  the mapping

$$A \mapsto \sigma A$$

is a linear mapping (homomorphism) of a vector space  $\mathcal{T}_p(\mathcal{V})$  onto itself. Moreover, as can easily be seen,

$$(\sigma\tau) A = \sigma(\tau A)$$

for any permutations  $\sigma, \tau \in S_p$ , from which in particular it follows that the mapping  $A \mapsto \sigma A$  is an isomorphism.  $\square$

From now on we shall assume that the ground field  $\mathbb{K}$  has the characteristic 0, i.e. it is possible to divide in it by any natural number (and in particular by the factorial  $p!$ ).

**Definition 3.** For any functional  $A \in \mathcal{T}_p(\mathcal{V})$  the symbol  $\text{Alt } A$  designates a functional defined by the formula

$$\text{Alt } A = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma (\sigma A).$$

It is clear that the mapping

$$A \mapsto \text{Alt } A$$

is linear (is a homomorphism). It is called an *alternation*.

Since  $\text{Alt}: \mathcal{T}_p(\mathcal{V}) \rightarrow \mathcal{T}_p(\mathcal{V})$  and  $\sigma: \mathcal{T}_p(\mathcal{V}) \rightarrow \mathcal{T}_p(\mathcal{V})$ , the composition mappings  $\sigma \circ \text{Alt}$  and  $\text{Alt} \circ \sigma$  are well-defined.

**Proposition 2.** For any permutation  $\sigma \in S_p$  there are relations

$$\text{Alt} \circ \sigma = \varepsilon_\sigma \text{Alt}, \quad \sigma \circ \text{Alt} = \varepsilon_\sigma \text{Alt}.$$

*Proof.* For every functional  $A \in \mathcal{T}_p(\mathcal{V})$  we have

$$\begin{aligned} \text{Alt}(\sigma A) &= \frac{1}{p!} \sum_{\tau \in S_p} \varepsilon_\tau (\tau \sigma A) = \\ &= \varepsilon_\sigma \frac{1}{p!} \sum_{\tau \in S_p} \varepsilon_{\tau\sigma} (\tau \sigma A) = \\ &= \varepsilon_\sigma \frac{1}{p!} \sum_{\tau \in S_p} \varepsilon_\tau (\tau A) = \varepsilon_\sigma \text{Alt } A, \end{aligned}$$

for  $\tau\sigma$  runs simultaneously with  $\tau$  over the whole group  $S_p$ . Similarly,

$$\sigma \text{Alt } A = \frac{1}{p!} \sum_{\tau \in S_p} \varepsilon_\tau (\sigma\tau A) = \varepsilon_\sigma \frac{1}{p!} \sum_{\tau \in S_p} \varepsilon_{\sigma\tau} (\sigma\tau A) = \varepsilon_\sigma \text{Alt } A. \quad \square$$

Since  $\text{Alt}: \mathbf{T}_p(\mathcal{V}) \rightarrow \mathbf{T}_p(\mathcal{V})$  the iteration

$$\text{Alt} \circ \text{Alt}: \mathbf{T}_p(\mathcal{V}) \rightarrow \mathbf{T}_p(\mathcal{V})$$

is well-defined.

**Proposition 3.** *The following equation holds*

$$\text{Alt} \circ \text{Alt} = \text{Alt}.$$

*Proof.* By linearity of alternation and Proposition 2

$$\begin{aligned} \text{Alt}(\text{Alt } A) &= \text{Alt} \left( \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma (\sigma A) \right) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma \text{Alt}(\sigma A) = \frac{1}{p!} \sum_{\sigma \in S_p} (\varepsilon_\sigma)^2 \text{Alt } A = \text{Alt } A. \quad \square \end{aligned}$$

How can the coefficients  $(\text{Alt } A)_{i_1 \dots i_p}$  of the functional  $\text{Alt } A$  be expressed in terms of the coefficients  $A_{i_1 \dots i_p}$  of the functional  $A$ ? It is appropriate to introduce a compact notation for these formulas that may be useful for other purposes.

Let  $A_{i_1 \dots i_p}$  be  $n^p$  given numbers with indices  $i_1, \dots, i_p$  varying from 1 to  $n$ . Associate with them other  $n^p$  numbers  $B_{i_1 \dots i_p}$  similarly indexed and defined by the formula

$$B_{i_1 \dots i_p} = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma A_{i_{\sigma(1)} \dots i_{\sigma(p)}}.$$

Allowing for a certain degree of inaccuracy in the formulas, numbers  $B_{i_1 \dots i_p}$  are usually denoted by  $A_{[i_1 \dots i_p]}$ . Thus by definition

$$A_{[i_1 \dots i_p]} = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma A_{i_{\sigma(1)} \dots i_{\sigma(p)}}.$$

Of course the position of *indices* does not play any role in this notation. If superscripts are used in denoting the given  $n^p$  numbers:  $A^{i_1 \dots i_p}$ , then one accordingly sets

$$A^{[i_1 \dots i_p]} = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_{\sigma} A^{i_{\sigma(1)} \dots i_{\sigma(p)}}.$$

**Proposition 4.** *For the coefficients of a functional Alt  $A$  the following formula holds*

$$(\text{Alt } A)_{i_1 \dots i_p} = A_{[i_1 \dots i_p]}.$$

*Proof.* By definition

$$\begin{aligned} (\text{Alt } A)_{i_1 \dots i_p} &= \text{Alt } A (\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_p}) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_{\sigma} (\sigma A) (\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_p}) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_{\sigma} A (\mathbf{e}_{i_{\sigma(1)}}, \dots, \mathbf{e}_{i_{\sigma(p)}}) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_{\sigma} A_{i_{\sigma(1)} \dots i_{\sigma(p)}} = A_{[i_1 \dots i_p]}. \quad \square \end{aligned}$$

The significance of the notation introduced above is not exhausted by this formula. For example, it is convenient to use it in writing determinants.

**Lemma 1.** *The following identity holds*

$$\begin{vmatrix} x_1^1 & \dots & x_1^p \\ x_2^1 & \dots & x_2^p \\ \dots & \dots & \dots \\ x_p^1 & \dots & x_p^p \end{vmatrix} = p! \, x_1^{[1] \dots [p]} = p! \, x_{[1] \dots [p]}^p.$$

*Proof.* By definition

$$p! \, x_1^{[1] \dots [p]} = \sum_{\sigma \in S_p} \varepsilon_{\sigma} x_1^{\sigma(1)} \dots x_p^{\sigma(p)}$$

and

$$p! \, x_{[1] \dots [p]}^p = \sum_{\sigma \in S_p} \varepsilon_{\sigma} x_{\sigma(1)}^1 \dots x_{\sigma(p)}^p.$$

In both cases the expression on the right is equal to the determinant  $|x_i^j|$  (first expanded “by the rows” and then “by the columns”).  $\square$

**Proposition 5.** *For any vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  the following formulas hold*

$$\begin{aligned} (\text{Alt } A)(\mathbf{x}_1, \dots, \mathbf{x}_p) &= A_{[i_1 \dots i_p]} x_1^{i_1} \dots x_p^{i_p} = \\ &= A_{i_1 \dots i_p} x_{[1}^{i_1} \dots x_p^{i_p]} = \\ &= A_{i_1 \dots i_p} x_1^{[i_1} \dots x_p^{i_p]}. \end{aligned}$$

*Proof.* The first formula is but a different way of writing the statement of Proposition 4. The second is proved by computation:

$$\begin{aligned} (\text{Alt } A)(\mathbf{x}_1, \dots, \mathbf{x}_p) &= \frac{1^n}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma A(x_{\sigma(1)}, \dots, x_{\sigma(p)}) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma A_{i_1 \dots i_p} x_{\sigma(1)}^{i_1} \dots x_{\sigma(p)}^{i_p} = \\ &= A_{i_1 \dots i_p} \left( \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma x_{\sigma(1)}^{i_1} \dots x_{\sigma(p)}^{i_p} \right) = \\ &= A_{i_1 \dots i_p} x_{[1}^{i_1} \dots x_p^{i_p]}. \end{aligned}$$

The third formula follows from the second by virtue of Lemma 1.  $\square$

**Corollary.** *The following formula holds*

$$(\text{Alt } A)(\mathbf{x}_1, \dots, \mathbf{x}_p) = \frac{1}{p!} A_{i_1 \dots i_p} \begin{vmatrix} x_1^{i_1} & \dots & x_p^{i_1} \\ x_1^{i_2} & \dots & x_p^{i_2} \\ \dots & \dots & \dots \\ x_1^{i_p} & \dots & x_p^{i_p} \end{vmatrix},$$

where as always summation over  $i_1, \dots, i_p$  is carried out from 1 to  $n$ ,

**Example.** For  $p = 2$

$$\begin{aligned}
 (\text{Alt } A)(\mathbf{x}, \mathbf{y}) &= \left( \frac{A_{ij} - A_{ji}}{2} \right) x^i y^j = \\
 &= A_{ij} \left( \frac{x^i y^j - y^i x^j}{2} \right) = \\
 &= A_{ij} \left( \frac{x^i y^j - x^j y^i}{2} \right) = \\
 &= \frac{1}{2} A_{ij} \begin{vmatrix} x^i & y^i \\ x^j & y^j \end{vmatrix}.
 \end{aligned}$$

# Lecture 8

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*Skew-symmetric multilinear functionals • External multiplication • Grassman algebra • External sums of covectors • Expansion of skew-symmetric functionals with respect to the external products of covectors of a basis*

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Suppose we are given two sets of  $n^p$  numbers  $x_{i_1 \dots i_p}$  and  $y^{i_1 \dots i_p}$  with  $p$  indices  $i_1, \dots, i_p$  independently running from 1 to  $n$ . On multiplying each of the numbers  $x_{i_1 \dots i_p}$  by a corresponding number  $y^{i_1 \dots i_p}$  and adding all the products together we obtain a number

$$x_{i_1 \dots i_p} y^{i_1 \dots i_p}.$$

**Lemma 1.** *For any permutation  $\sigma \in S_p$  we have the identity*

$$(1) \quad x_{i_1 \dots i_p} y^{i_1 \dots i_p} = x_{i_{\sigma(1)} \dots i_{\sigma(p)}} y^{i_{\sigma(1)} \dots i_{\sigma(p)}}.$$

*Proof.* Both sides of relation (1) are sums of the same but differently ordered terms.  $\square$

Suppose we are given  $n^2$  numbers  $x_j^i$ , where  $i, j = 1, \dots, n$ . Consider all possible products of the form

$$x_{j_1}^{i_1} \dots x_{j_p}^{i_p}.$$

**Lemma 2.** *For any permutation  $\sigma \in S_p$  we have the identity*

$$(2) \quad x_{j_1}^{i_{\sigma(1)}} \dots x_{j_p}^{i_{\sigma(p)}} = x_{j_{\tau(1)}}^{i_1} \dots x_{j_{\tau(p)}}^{i_p},$$

where  $\tau = \sigma^{-1}$ .

*Proof.* Both sides of relation (2) are products of the same but differently ordered multipliers. For example, if  $p = 4$  and

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 3 & 1 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 3 & 2 \end{pmatrix},$$

then

$$x_{j_1}^{i_2} x_{j_2}^{i_4} x_{j_3}^{i_3} x_{j_4}^{i_1} = x_{j_4}^{i_1} x_{j_1}^{i_2} x_{j_3}^{i_3} x_{j_2}^{i_4}. \quad \square$$

**Definition 1.** A multilinear functional  $A$  is said to be *skew-symmetric* if

$$\sigma A = \varepsilon_\sigma A$$

for any permutation  $\sigma \in S_p$ .

**Proposition 1.** A functional  $A$  is skew-symmetric if and only if

$$(3) \quad A_{i_{\sigma(1)} \dots i_{\sigma(p)}} = \varepsilon_\sigma A_{i_1 \dots i_p}$$

for any indices  $i_1, \dots, i_p$  and any permutation  $\sigma \in S_p$ .

*Proof.* If  $A$  is skew-symmetric, then

$$A_{i_{\sigma(1)} \dots i_{\sigma(p)}} = (\sigma A)_{i_1 \dots i_p} = \varepsilon_\sigma A_{i_1 \dots i_p}.$$

Conversely, if relation (3) holds, then for any vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  we have

$$(\sigma A)(\mathbf{x}_1, \dots, \mathbf{x}_p) = A(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(p)}) = A_{i_1 \dots i_p} x_{\sigma(1)}^{i_1} \dots x_{\sigma(p)}^{i_p}.$$

But according to Lemmas 1 and 2 and condition (3)

$$\begin{aligned} A_{i_1 \dots i_p} x_{\sigma(1)}^{i_1} \dots x_{\sigma(p)}^{i_p} &= A_{i_{\sigma(1)} \dots i_{\sigma(p)}} x_{\sigma(1)}^{i_{\sigma(1)}} \dots x_{\sigma(p)}^{i_{\sigma(p)}} = \\ &= A_{i_{\sigma(1)} \dots i_{\sigma(p)}} x_1^{i_1} \dots x_p^{i_p} = \\ &= \varepsilon_\sigma A_{i_1 \dots i_p} x_1^{i_1} \dots x_p^{i_p} = \\ &= \varepsilon_\sigma A(\mathbf{x}_1, \dots, \mathbf{x}_p). \end{aligned}$$

Hence  $\sigma A = \varepsilon_\sigma A$ .  $\square$



**Proposition 2.** *A multilinear functional  $A$  is skew-symmetric if and only if it remains unchanged when alternated*

$$\text{Alt } A = A.$$

*Proof.* If

$$\sigma A = \varepsilon_\sigma A$$

for any permutation  $\sigma \in S_p$ , then the terms of the sum

$$\sum_{\sigma \in S_p} \varepsilon_\sigma (\sigma A)$$

are all equal to  $A$ , and therefore this sum is equal to  $p!A$ . Hence

$$\text{Alt } A = A.$$

Conversely, if  $\text{Alt } A = A$ , then according to Proposition 2 of the preceding lecture

$$\sigma A = \sigma \text{Alt } A = \varepsilon_\sigma \text{Alt } A = \varepsilon_\sigma A. \quad \square$$

**Corollary.** *A multilinear functional  $A$  is skew-symmetric if and only if for its coefficients the following equations hold*

$$A_{i_1 \dots i_p} = A_{[i_1 \dots i_p]}.$$

A formally somewhat more general condition of the skew-symmetry of a functional is given by the following proposition:

**Proposition 3.** *A multilinear functional  $A$  is skew-symmetric if and only if there exists a multilinear functional  $B$  such that*

$$(4) \quad A = \text{Alt } B.$$

*Proof.* If  $A$  is skew-symmetric, then (4) holds for  $B = A$  (Proposition 2). Conversely, if (4) holds, then according to Proposition 3 of the preceding lecture

$$\text{Alt } A = \text{Alt } (\text{Alt } B) = \text{Alt } B = A$$

and hence (Proposition 2)  $A$  is skew-symmetric.  $\square$

A tensor product  $A \otimes B$  of two skew-symmetric functionals will not in general be a skew-symmetric functional. To turn this product into a skew-symmetric functional it is necessary to alternate it.

**Definition 2.** An external product  $A \wedge B$  of skew-symmetric functionals  $A$  and  $B$  is the functional

$$A \wedge B = \text{Alt} (A \otimes B).$$

Its degree is equal to  $p + q$ , where  $p$  and  $q$  are the degrees of  $A$  and  $B$ , and its coefficients are expressed by the formula

$$(A \wedge B)_{i_1 \dots i_{p+q}} = A_{[i_1 \dots i_p} B_{i_{p+1} \dots i_{p+q}]}.$$

**Proposition 4.** External multiplication of skew-symmetric functionals is associative, i.e.

$$(A \wedge B) \wedge C = A \wedge (B \wedge C)$$

for any three skew-symmetric functionals  $A$ ,  $B$  and  $C$ .

By virtue of this proposition one may omit brackets in the external products of several functionals.

We shall preface the proof of Proposition 4 with some remarks that are of interest in themselves.

For any  $p$  and  $q$  we can map a symmetric group  $S_p$  into a symmetric group  $S_{p+q}$  by associating with an arbitrary permutation  $\sigma \in S_p$  a permutation  $\sigma' \in S_{p+q}$  acting on the numbers  $1, \dots, p$  in the same way as  $\sigma$  and leaving the numbers  $p + 1, \dots, q$  fixed:

$$\sigma' (i) = \begin{cases} \sigma (i), & 1 \leq i \leq p, \\ i, & p + 1 \leq i \leq p + q. \end{cases}$$

It is clear that the correspondence  $\sigma \mapsto \sigma'$  is a monomorphism (an injective homomorphism) preserving the sign, i.e. such that

$$\varepsilon_{\sigma'} = \varepsilon_{\sigma}$$

for any permutation  $\sigma$ .

Applying permutations of the form  $\sigma \in S_p$  it is possible to have an arbitrary multilinear functional  $A$  of degree  $p + q$  "alternated only by the first  $p$  arguments" i.e. to construct a functional

$$\text{alt } A = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_{\sigma} (\sigma' A).$$

**Lemma 3.**  $\text{Alt}(\text{alt } A) = \text{Alt } A$ .

The proof of this lemma actually completely repeats that of Proposition 3 of the preceding lecture:

$$\begin{aligned} \text{Alt}(\text{alt } A) &= \text{Alt} \left( \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma (\sigma' A) \right) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma \varepsilon_{\sigma'} \text{Alt } A = \text{Alt } A. \quad \square \end{aligned}$$

We can now pass directly on to the proof of Proposition 4.

*Proof of Proposition 4.* Let  $p$ ,  $q$ , and  $r$  be the degrees of the functionals  $A$ ,  $B$  and  $C$ .

By definition

$$(A \wedge B) \wedge C = \text{Alt}((A \wedge B) \otimes C).$$

But it is clear that

$$(A \wedge B) \otimes C = \text{alt}(A \otimes B \otimes C),$$

where  $\text{alt}$  designates alternation by the first  $p + q$  indices. Therefore according to Lemma 3

$$(A \wedge B) \wedge C = \text{Alt}(A \otimes B \otimes C).$$

We can similarly prove that

$$A \wedge (B \wedge C) = \text{Alt}(A \otimes B \otimes C).$$

Consequently  $(A \wedge B) \wedge C = A \wedge (B \wedge C)$ .  $\square$

We see in particular that

$$A \wedge B \wedge C = \text{Alt}(A \otimes B \otimes C).$$

It is clear that a similar formula holds for any number of multipliers.

Unlike tensor multiplication, external multiplication is commutative, although up to a sign.

**Proposition 5.** *For any two skew-symmetric multilinear functionals  $A$  and  $B$  of degrees  $p$  and  $q$  the following equation holds*

$$B \wedge A = (-1)^{pq} A \wedge B.$$

This property of external multiplication is called *skew-commutativity*.

*Proof.* By definition

$$\begin{aligned}
 (B \otimes A)(\mathbf{x}_1, \dots, \mathbf{x}_{p+q}) &= B(\mathbf{x}_1, \dots, \mathbf{x}_q) A(\mathbf{x}_{q+1}, \dots, \mathbf{x}_{p+q}) = \\
 &= A(\mathbf{x}_{q+1}, \dots, \mathbf{x}_{p+q}) B(\mathbf{x}_1, \dots, \mathbf{x}_q) = \\
 &= (A \otimes B)(\mathbf{x}_{q+1}, \dots, \mathbf{x}_{p+q}, \mathbf{x}_1, \dots, \mathbf{x}_q) = \\
 &= (\sigma_0(A \otimes B))(\mathbf{x}_1, \dots, \mathbf{x}_{p+q}),
 \end{aligned}$$

where

$$\sigma_0 = \begin{pmatrix} 1, \dots, p & p+1, \dots, p+q \\ q+1, \dots, p+q, & 1, \dots, q \end{pmatrix},$$

i.e.

$$B \otimes A = \sigma_0(A \otimes B).$$

Therefore

$$B \wedge A = \text{Alt}(B \otimes A) = \varepsilon_{\sigma_0} \text{Alt}(A \otimes B) = \varepsilon_{\sigma_0}(A \wedge B).$$

To complete the proof it remains to note that

$$\varepsilon_{\sigma_0} = (-1)^{pq}. \quad \square$$

It is clear that the set  $\Lambda_p(\mathcal{V})$  of all skew-symmetric functionals of degree  $p$  is a subspace of the space  $\mathsf{T}_p(\mathcal{V})$  and hence is itself a vector space. The operation of external multiplication of skew-symmetric functionals is obviously distributive over addition:

$$(A + B) \wedge C = A \wedge C + B \wedge C.$$

This means that under  $+$  and  $\wedge$  operations the vector spaces

$$\Lambda_0(\mathcal{V}), \Lambda_1(\mathcal{V}), \dots, \Lambda_p(\mathcal{V}), \dots$$

constitute an algebraic object that is an example of what is called *graded algebra*. This is designated by the symbol  $\Lambda(\mathcal{V})$  and called *the exterior algebra* of a space  $\mathcal{V}$  (or its *Grassman algebra*).

Note that for  $p = 1$  the skew-symmetry condition imposes no restrictions. Therefore

$$\Lambda_1(\mathcal{V}) = \mathsf{T}_1(\mathcal{V}) = \mathcal{V}'.$$

By similar considerations

$$\Lambda_0(\mathcal{V}) = \mathsf{T}_0(\mathcal{V}) = \mathbb{K}.$$

It follows from skew-symmetry in an obvious way that  $A(\mathbf{x}_1, \dots, \mathbf{x}_p) = 0$  if at least two vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  coincide (recall the corresponding reasoning for determinants). Hence, by multilinearity,  $A(\mathbf{x}_1, \dots, \mathbf{x}_p) = 0$  if one of the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  is linearly expressible in terms of the others. Since for  $p > n$  this is always the case, we thus get

$$\Lambda_p(\mathcal{V}) = 0 \quad \text{for } p > n.$$

Of particular interest are external products of first-degree functionals, i.e. of covectors.

According to the remark made above

$$\xi^1 \wedge \dots \wedge \xi^p = \text{Alt}(\xi^1 \otimes \dots \otimes \xi^p)$$

for any covectors  $\xi^1, \dots, \xi^p$ . This means that for any vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  we have

$$\begin{aligned} (\xi^1 \wedge \dots \wedge \xi^p)(\mathbf{x}_1, \dots, \mathbf{x}_p) &= \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma (\xi^1 \otimes \dots \otimes \xi^p)(\mathbf{x}_{\sigma(1)}, \dots, \mathbf{x}_{\sigma(p)}) = \\ &= \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma \xi^1(\mathbf{x}_{\sigma(1)}) \dots \xi^p(\mathbf{x}_{\sigma(p)}), \end{aligned}$$

i.e.

$$(5) \quad (\xi^1 \wedge \dots \wedge \xi^p)(\mathbf{x}_1, \dots, \mathbf{x}_p) = \frac{1}{p!} \begin{vmatrix} \xi^1(\mathbf{x}_1) & \dots & \xi^1(\mathbf{x}_p) \\ \vdots & & \vdots \\ \xi^p(\mathbf{x}_1) & \dots & \xi^p(\mathbf{x}_p) \end{vmatrix}.$$

This—very important!—identity can be rewritten (see Lemma 1 of the preceding lecture) in the following equivalent form

$$(\xi^1 \wedge \dots \wedge \xi^p)(\mathbf{x}_1, \dots, \mathbf{x}_p) = \xi^1(\mathbf{x}_{[1]} \dots \xi^p(\mathbf{x}_{[p]}),$$

or in the form

$$(6) \quad (\xi^1 \wedge \dots \wedge \xi^p)(\mathbf{x}_1, \dots, \mathbf{x}_p) = \xi^{[1]}(\mathbf{x}_1) \dots \xi^{[p]}(\mathbf{x}_p).$$

We now introduce the functional

$$\xi^{[1]} \otimes \dots \otimes \xi^{[p]} = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma \xi^{\sigma(1)} \otimes \dots \otimes \xi^{\sigma(p)}.$$

Its value on the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  is expressed by the formula

$$\begin{aligned}
 (7) \quad & (\xi^{[1} \otimes \dots \otimes \xi^{p]}) (\mathbf{x}_1, \dots, \mathbf{x}_p) = \\
 & = \frac{1}{p!} \sum_{\sigma \in S_p} \varepsilon_\sigma (\xi^{\sigma(1)} \otimes \dots \otimes \xi^{\sigma(p)}) (\mathbf{x}_1, \dots, \mathbf{x}_p) = \\
 & = \frac{1}{p!} \sum_{\sigma \in S_p} \xi_{\sigma(1)}^{\sigma(1)} (\mathbf{x}_1) \dots \xi_{\sigma(p)}^{\sigma(p)} (\mathbf{x}_p) = \xi^{[1} (\mathbf{x}_1) \dots \xi^{p]} (\mathbf{x}_p).
 \end{aligned}$$

Comparing formulas (6) and (7) we obtain the following proposition:

**Proposition 6.** *For any covectors  $\xi^1, \dots, \xi^p$  we have*

$$\xi^1 \wedge \dots \wedge \xi^p = \xi^{[1} \otimes \dots \otimes \xi^{p]}. \square$$

**Corollary.** *A functional  $\xi^1 \wedge \dots \wedge \xi^p$  is in tensor form expressible in terms of covectors  $\xi^1, \dots, \xi^p$ .  $\square$*

We now prove a simple but important proposition.

**Proposition 7.** *The equation*

$$\xi^1 \wedge \dots \wedge \xi^p = 0$$

*holds if and only if the covectors  $\xi^1, \dots, \xi^p$  are linearly dependent.*

*Proof.* By skew-commutativity of external multiplication the product  $\xi^1 \wedge \dots \wedge \xi^p$  changes the sign when any two multipliers are interchanged. By a now familiar reasoning it can be deduced from this that  $\xi^1 \wedge \dots \wedge \xi^p = 0$  if the covectors  $\xi^1, \dots, \xi^p$  are linearly dependent.

Let the covectors  $\xi^1, \dots, \xi^p$  be linearly independent. Then they can be supplemented to obtain a basis

$$\mathbf{e}^1 = \xi^1, \dots, \mathbf{e}^p = \xi^p, \mathbf{e}^{p+1}, \dots, \mathbf{e}^n$$

of the whole space  $\mathcal{V}'$ . Let

$$\mathbf{e}_1, \dots, \mathbf{e}_n$$

be a conjugate basis of a space  $\mathcal{V}$ . Then according to formula (5)

$$\begin{aligned}
 (\xi^1 \wedge \dots \wedge \xi^p)(\mathbf{e}_1, \dots, \mathbf{e}_p) &= \frac{1}{p!} \begin{vmatrix} \xi^1(\mathbf{e}_1) & \dots & \xi^1(\mathbf{e}_p) \\ \dots & \dots & \dots \\ \xi^p(\mathbf{e}_1) & \dots & \xi^p(\mathbf{e}_p) \end{vmatrix} = \\
 &= \frac{1}{p!} \begin{vmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{vmatrix} = \frac{1}{p!} \neq 0.
 \end{aligned}$$

Therefore  $\xi^1 \wedge \dots \wedge \xi^p \neq 0$ .  $\square$

Let  $\mathbf{e}^1, \dots, \mathbf{e}^n$  be an arbitrary basis of a space  $\mathcal{V}'$ . Then every multilinear functional  $A$  allows, as we know, a representation of the form

$$A = A_{i_1 \dots i_p} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p}.$$

If the functional is skew-symmetric (and hence  $A = \text{Alt } A$ ), then after alternating we obtain from this a formula of the form

$$(8) \quad A = A_{i_1 \dots i_p} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p}.$$

There are, however, many zero and identical terms in this formula. We should therefore “reduce similar terms” in it.

According to Proposition 7 the terms in the sum (8) for which there are identical indices among the indices  $i_1, \dots, i_p$  are all equal to zero. Therefore

$$(9) \quad A = \sum A_{i_1 \dots i_p} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p},$$

where summation is taken over all  $p$ -member sets  $i_1, i_2, \dots, i_p$  of integers 1 to  $n$  consisting of *different* numbers.

On fixing one of such sets consider in the sum (9) the terms differing only in the order of their indices. There are  $p!$  such terms in all and each has the form

$$(10) \quad \underbrace{A_{i_{\sigma(1)} \dots i_{\sigma(p)}} \mathbf{e}^{i_{\sigma(1)}} \wedge \dots \wedge \mathbf{e}^{i_{\sigma(p)}}}_{\text{(without summation)}},$$

where  $\sigma$  is an arbitrary permutation of degree  $p$ . But, as is immediate from the skew-commutativity of external multiplication

$$e^{i_{\sigma(1)}} \wedge \dots \wedge e^{i_{\sigma(p)}} = \varepsilon_{\sigma} e^{i_1} \wedge \dots \wedge e^{i_p}.$$

On the other hand, according to Proposition 1

$$A_{i_{\sigma(1)} \dots i_{\sigma(p)}} = \varepsilon_{\sigma} A_{i_1 \dots i_p}.$$

Since  $\varepsilon_{\sigma} \varepsilon_{\sigma} = 1$ , all of the terms (10) are equal to

$$\underbrace{A_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}}_{\text{(without summation)}}$$

This proves that

$$A = p! \sum_{(i_1, \dots, i_p)} A_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p},$$

where summation is taken over all *combinations*  $(i_1, \dots, i_p)$  of indices in the sum (9). Since for every combination there exists a unique set  $i_1, \dots, i_p$  for which  $i_1 < i_2 < \dots < i_p$ , this proves the following proposition:

**Proposition 8.** *For any skew-symmetric functional  $A$  the following equation holds*

$$A = p! \sum_{i_1 < \dots < i_p} A_{i_1 \dots i_p} e^{i_1} \wedge \dots \wedge e^{i_p}. \quad \square$$

Thus functionals of the form

$$e^{i_1} \wedge \dots \wedge e^{i_p}, \quad 1 \leq i_1 < \dots < i_p \leq n,$$

constitute a family complete in  $\Lambda_p(\mathcal{V})$ .



# Lecture 9

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*The basis of a space of skew-symmetric functionals • Formulas for the transformation of the basis of that space • Multivectors • The external rank of a skew-symmetric functional • Multivector rank theorem • Conditions for the equality of multivectors*

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It follows from Proposition 1 of the preceding lecture that for the coefficients  $A_{i_1 \dots i_p}$  of a skew-symmetric functional  $A$  we have the equations

$$(1) \quad A_{i_1 \dots i_p} = \begin{cases} 0 & \text{if there are identical numbers} \\ & \text{among the numbers } i_1, \dots, i_p, \\ \varepsilon_\sigma A_{i_{\sigma(1)} \dots i_{\sigma(p)}} & \text{otherwise,} \end{cases}$$

where  $\sigma$  is a permutation of degree  $p$  such that

$$i_{\sigma(1)} < \dots < i_{\sigma(p)}.$$

It follows that in order to completely reconstruct the functional  $A$  it is sufficient to know only those of its coefficients  $A_{i_1 \dots i_p}$  for which  $i_1 < \dots < i_p$ .

**Definition 1.** The coefficients  $A_{i_1 \dots i_p}$  for which  $i_1 < \dots < i_p$  are called *the essential coefficients* of a skew-symmetric functional  $A$ .

**Proposition 1.** For any  $\binom{n}{p}$  numbers

$$(2) \quad A_{i_1 \dots i_p}$$

with indices  $i_1 < \dots < i_p$  there exists a unique skew-symmetric functional  $A$  the essential coefficients of which are these numbers.

*Proof.* The uniqueness of the functional  $A$  has just been established. We should therefore prove only its existence. On determining  $n^p$  numbers  $A_{i_1 \dots i_p}$  for all  $i_1, \dots, i_p$  by means of formulas (1) consider a multilinear functional

$$(3) \quad A = A_{i_1 \dots i_p} e^{i_1} \otimes \dots \otimes e^{i_p}.$$

It is clear that if the functional is skew-symmetric, then its essential coefficients are precisely the numbers (2). Everything will thus be proved if we show that the functional (3) is skew-symmetric.

To do this it suffices, according to Proposition 1, to prove that for the coefficients of the functional (3) we have the relations

$$(4) \quad A_{i_{\sigma(1)} \dots i_{\sigma(p)}} = \varepsilon_{\sigma} A_{i_1 \dots i_p},$$

where  $\sigma$  is an arbitrary permutation of degree  $p$ . And we can obviously assume without loss of generality that the indices  $i_1, \dots, i_p$  are all different (since otherwise both sides of formula (4) are equal to zero).

But if the indices  $i_1, \dots, i_p$  are different, then by definition

$$A_{i_1 \dots i_p} = \varepsilon_{\tau} A_{i_{\tau(1)} \dots i_{\tau(p)}}$$

where  $\tau$  is a permutation such that  $i_{\tau(1)} < \dots < i_{\tau(p)}$ . Similarly

$$A_{i_{\sigma(1)} \dots i_{\sigma(p)}} = \varepsilon_{\rho} A_{i_{\rho(\sigma(1))} \dots i_{\rho(\sigma(p))}},$$

where  $\rho$  is a permutation such that  $i_{\rho(\sigma(1))} < \dots < i_{\rho(\sigma(p))}$ . But the numbers  $i_{\tau(1)}, \dots, i_{\tau(p)}$  and  $i_{\rho(\sigma(1))}, \dots, i_{\rho(\sigma(p))}$  are the same, since both the former and the latter are the indices  $i_1, \dots, i_p$  arranged in the order of increasing. Consequently

$$\tau(1) = \rho(\sigma(1)), \dots, \tau(p) = \rho(\sigma(p)),$$

i.e.  $\tau = \rho\sigma$ . Therefore  $\varepsilon_{\tau} = \varepsilon_{\rho}\varepsilon_{\sigma}$  and hence

$$A_{i_1 \dots i_p} = \varepsilon_{\sigma} A_{i_{\sigma(1)} \dots i_{\sigma(p)}}. \quad \square$$

**Theorem 1.** *The external products*

$$(5) \quad e^{i_1} \wedge \dots \wedge e^{i_p}, \quad 1 \leq i_1 < \dots < i_p \leq n$$

constitute a basis of a space  $\Lambda_p(\mathcal{V})$ .

*Proof.* In view of Proposition 8 of the preceding lecture it is sufficient to prove that the functionals (5) are linearly independent.

Let

$$\sum_{i_1 < \dots < i_p} A_{i_1 \dots i_p} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p} = 0,$$

where  $A_{i_1 \dots i_p}$ ,  $i_1 < \dots < i_p$  are some numbers. According to Proposition 1 there exists a skew-symmetric functional  $A$  the essential coefficients of which are the numbers  $A_{i_1 \dots i_p}$ . According to Proposition 8 of the preceding lecture that functional can be expressed by the formula

$$A = \frac{1}{p!} \sum_{i_1 < \dots < i_p} A_{i_1 \dots i_p} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p}$$

and hence is under the hypothesis equal to zero. But then

$$A_{i_1 \dots i_p} = A(\mathbf{e}_{i_1}, \dots, \mathbf{e}_{i_p}) = 0.$$

Therefore the functionals (5) are linearly independent.  $\square$

**Corollary 1.** *The representation of a skew-symmetric functional  $A$  as*

$$A = \frac{1}{p!} \sum_{i_1 < \dots < i_p} A_{i_1 \dots i_p} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p}$$

*is unique.*  $\square$

**Corollary 2.** *The dimension of a space  $\Lambda_p(\mathcal{V})$  is equal to  $\binom{n}{p}$ :*

$$(6) \quad \dim \Lambda_p(\mathcal{V}) = \binom{n}{p}. \quad \square$$

In particular we again see that

$$\Lambda_p(\mathcal{V}) = 0 \quad \text{for } p > n.$$

Let us transform from the basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  to another basis:  $\mathbf{e}^{1'}, \dots, \mathbf{e}^{n'}$ . If as always

$$\mathbf{e}^{i'} = c_i^{i'} \mathbf{e}^i,$$

then

$$\mathbf{e}^{i'_1} \otimes \dots \otimes \mathbf{e}^{i'_p} = c_{i_1}^{i'_1} \dots c_{i_p}^{i'_p} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p}.$$

Therefore (see Proposition 4 of Lecture 7)

$$\begin{aligned} \mathbf{e}^{i'_1} \wedge \dots \wedge \mathbf{e}^{i'_p} &= \text{Alt} (\mathbf{e}^{i'_1} \otimes \dots \otimes \mathbf{e}^{i'_p}) = \\ &= c_{[i_1}^{i'_1} \dots c_{i_p]}^{i'_p} \mathbf{e}^{i_1} \otimes \dots \otimes \mathbf{e}^{i_p}, \end{aligned}$$

and hence

$$\mathbf{e}^{i'_1} \wedge \dots \wedge \mathbf{e}^{i'_p} = p! \sum_{i_1 < \dots < i_p} c_{[i_1}^{i'_1} \dots c_{i_p]}^{i'_p} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p}.$$

The number  $p! c_{[i_1}^{i'_1} \dots c_{i_p]}^{i'_p}$  is equal (see Lemma 1 of the preceding lecture) to the minor

$$C \begin{pmatrix} i'_1 & \dots & i'_p \\ i_1 & \dots & i_p \end{pmatrix}$$

of the transition matrix  $C = (c_i^{i'})$  which is in the intersection of the columns with the numbers  $i_1 < \dots < i_p$  and the rows with the numbers  $i'_1 < \dots < i'_p$ . We can therefore write the obtained formula for the transformation of the bases of the space  $\Lambda_p(\mathcal{V})$  in the following final form:

$$(7) \quad \mathbf{e}^{i'_1} \wedge \dots \wedge \mathbf{e}^{i'_p} = \sum_{i_1 < \dots < i_p} C \begin{pmatrix} i'_1 & \dots & i'_p \\ i_1 & \dots & i_p \end{pmatrix} \mathbf{e}^{i_1} \wedge \dots \wedge \mathbf{e}^{i_p},$$

where  $i'_1 < \dots < i'_p$ .

The results obtained can all be transferred in a natural way to  $(0, p)$ -tensors, i.e. to multilinear functionals  $A: \xi^1, \dots, \xi^p \mapsto A(\xi^1, \dots, \xi^p)$  of degree  $p$  of covectors. The only difference is that the subscripts become superscripts and vice versa. In particular the coefficients of a  $(0, p)$ -functional  $A$  have the form  $A^{i_1 \dots i_p}$ , the basis of the space  $\Lambda^p(\mathcal{V}) = \Lambda_p(\mathcal{V}')$  of all skew-symmetric  $(0, p)$ -functionals consists of external products

$$\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p} = \text{Alt} (\mathbf{e}_{i_1} \otimes \dots \otimes \mathbf{e}_{i_p}),$$

where  $i_1 < \dots < i_p$ ,

and the expansion of an arbitrary skew-symmetric functional with respect to this basis is given by the formula

$$A = p! \sum_{i_1 < \dots < i_p} A^{i_1 \dots i_p} \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p}.$$





are bases of the same  $p$ -dimensional subspace. Therefore according to formula (10)

$$(13) \quad \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_p = \Delta (\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p),$$

where  $\Delta$  is the determinant (12). It remains to note that under the hypothesis  $\Delta = 1$ .  $\square$

The converse is significantly subtler. We shall preface its proof with some preliminaries.

For skew-symmetric functionals, as well as for arbitrary multilinear functionals, the concepts of rank and rank space are defined. But of course for such functionals the analogues of these concepts making use of external multiplication instead of tensor multiplication are much more natural.

Let  $A$  be an arbitrary skew-symmetric functional of degree  $p$  (of covectors, for definiteness).

**Definition 3.** A functional  $A$  is said to be *externally expressible* in terms of vectors  $\mathbf{x}_1, \dots, \mathbf{x}_r$  if it is a linear combination of external products  $\mathbf{x}_{i_1} \wedge \dots \wedge \mathbf{x}_{i_p}$ , where  $1 \leq i_1, \dots, i_p \leq r$ . Cf. Definition 4 of Lecture 6.

The number  $r$  is said to be *the external rank* of a skew-symmetric functional  $A$  if it satisfies the following conditions:

- (i) there exists a family of vectors consisting of  $r$  vectors in terms of which the functional  $A$  is externally expressible;
- (ii) in the case where the functional  $A$  is externally expressible in terms of some family of vectors the number of vectors in that family is not less than  $r$ .

It remarkably turns out, however, that these definitions are actually unnecessary since in fact *a skew-symmetric functional  $A$  is externally expressible in terms of vectors  $\mathbf{x}_1, \dots, \mathbf{x}_r$  if and only if it is expressible in terms of them in tensor form (so that the external rank of the functional  $A$  really coincides with its rank)*. Indeed, if

$$A = a^{i_1 \dots i_p} \mathbf{x}_{i_1} \otimes \dots \otimes \mathbf{x}_{i_p},$$

then alternating this equation we get

$$(14) \quad A = a^{i_1 \dots i_p} \mathbf{x}_{i_1} \wedge \dots \wedge \mathbf{x}_{i_p}.$$

Conversely, if the last equation holds, then according to formula (8)

$$A = a^{i_1 \dots i_p} \mathbf{x}_{[i_1} \otimes \dots \otimes \mathbf{x}_{i_p]}. \quad \square$$

Nevertheless the concept of external rank is not useless. It is clear indeed that the external rank of a nonzero skew-symmetric functional cannot be lower than its degree (for otherwise each term of the sum (14) would contain recurring multipliers). The same statement is true therefore also for the rank of the functional:

**Proposition 2.** *The rank  $r$  of a nonzero skew-symmetric functional  $A \in \Lambda^p(\mathcal{V})$  is not lower than its degree:*

$$p \leq r. \quad \square$$

We shall employ this important property many a time in what follows.

As a first application we shall prove the following statement characterizing multivectors in the class of all skew-symmetric functionals:

**Proposition 3.** *A skew-symmetric functional  $A \in \Lambda^p(\mathcal{V})$  is a multivector if and only if its rank  $r$  is equal to its degree:*

$$p = r.$$

*Proof.* If  $A = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$ , then obviously  $r \leq p$ . Therefore the equation  $r = p$  must hold in view of Proposition 2.

Conversely, if  $r = p$  then the functional  $A$  has the form

$$A = a^{i_1 \dots i_p} \mathbf{x}_{i_1} \wedge \dots \wedge \mathbf{x}_{i_p},$$

where  $\mathbf{x}_1, \dots, \mathbf{x}_p$  is a basis of its rank space. But then, as shown by the reasoning already repeatedly used above, the functional  $A$  is expressed by the formula

$$A = a^{[1 \dots p]} (\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p)$$

and hence by the formula

$$A = \mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_p,$$

where  $\mathbf{y}_1 = a^{[1 \dots p]} \mathbf{x}_1$ ,  $\mathbf{y}_2 = \mathbf{x}_2$ ,  $\dots$ ,  $\mathbf{y}_p = \mathbf{x}_p$ .  $\square$

**Corollary.** *The rank space of a multivector  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p \neq 0$  is the linear span  $[\mathbf{x}_1, \dots, \mathbf{x}_p]$  of the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$ .*

We can now find the equality conditions for two multivectors.





The equivalence of conditions (b) and (c) is obvious (and was already noted by us in Lecture 1).

If (c) holds, then the multivectors  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$  and  $\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_p$  are connected by relation (13) and hence (a) holds.

Finally, since the basis of the rank space of the functional  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$  consists of the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  (d) implies (c).

Consequently, conditions (a), (b), (c) and (d) are all equivalent.  $\square$

**Corollary.** *There is a natural bijective correspondence between classes of proportional nonzero  $p$ -vectors and  $p$ -dimensional subspaces of a space  $\mathcal{V}$ . In this correspondence to each subspace  $\mathcal{P}$  there corresponds an external product  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$  of vectors of its basis  $\mathbf{x}_1, \dots, \mathbf{x}_p$  and to each  $p$ -vector  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$  there corresponds a subspace  $[\mathbf{x}_1, \dots, \mathbf{x}_p]$ .*

Theorem 2 can now be proved without difficulty.

**Proof of Theorem 2.** We have already proved that if there hold relations (11) together with equation (12) then  $\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_p = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$ . Conversely, if  $\mathbf{y}_1 \wedge \dots \wedge \mathbf{y}_p = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$  then according to Proposition 4 there hold relations (11) and hence equation (13) with  $\Delta = 1$ .  $\square$

# Lecture 10

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*Cartan's divisibility theorem • Plücker relations • The Plücker coordinates of subspaces • Planes in an affine space • Planes in a projective space and their coordinates*

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The criterion established by Proposition 3 of the preceding lecture that the skew-symmetric functional is a multivector is ineffective in practice. To obtain a more convenient criterion it is necessary to previously prove the following statement known as *E. Cartan's theorem on divisibility* (the divisibility of a skew-symmetric functional by a multi-vector is implied).

**Proposition 1.** *Let  $\mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r \neq 0$ . For a skew-symmetric functional  $A$  of degree  $p \geq r$ , there is a skew-symmetric functional  $B$  of degree  $p - r$  such that*

$$A = B \wedge \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r,$$

if and only if

$$(1) \quad A \wedge \mathbf{x}_1 = 0, \dots, A \wedge \mathbf{x}_r = 0.$$

*Proof.* If  $A = B \wedge \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r$ , then for any  $s = 1, \dots, r$  the external product

$$A \wedge \mathbf{x}_s = B \wedge \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r \wedge \mathbf{x}_s$$

contains two multipliers  $\mathbf{x}_s$  and is therefore zero.

Conversely let relations (1) hold. Since the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_r$  are under the hypothesis linearly independent, they can be supplemented to form some basis

$$\mathbf{e}_1 = \mathbf{x}_1, \dots, \mathbf{e}_r = \mathbf{x}_r, \mathbf{e}_{r+1}, \dots, \mathbf{e}_n$$

of a space  $\mathcal{V}$ . Let

$$(2) \quad A = p! \sum_{1 \leq i_1 < \dots < i_p \leq n} A^{i_1 \dots i_p} \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p}$$

be an expansion of the functional  $A$  with respect to the corresponding basis of a space  $\Lambda_p(\mathcal{V})$  (see Theorem 1 of the preceding lecture). Then for any  $s = 1, \dots, r$

$$A \wedge \mathbf{x}_s = p! \sum_{1 \leq i_1 < \dots < i_p \leq n} A^{i_1 \dots i_p} \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p} \wedge \mathbf{e}_s.$$

If  $s$  is equal to one of the numbers  $i_1, \dots, i_p$ , then  $\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p} \wedge \mathbf{e}_s = 0$ . In the sum for  $A \wedge \mathbf{x}_s$  we therefore may restrict ourselves to the terms for which all indices  $i_1, \dots, i_p$  are other than  $s$ :

$$(3) \quad A \wedge \mathbf{x}_s = p! \sum_{\substack{i_1 < \dots < i_p \\ i_1, \dots, i_p \neq s}} A^{i_1 \dots i_p} \mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p} \wedge \mathbf{e}_s.$$

But when  $i_1, \dots, i_p$  are not equal to  $s$ , all multivectors of the form

$$\mathbf{e}_{i_1} \wedge \dots \wedge \mathbf{e}_{i_p} \wedge \mathbf{e}_s, \quad 1 \leq i_1 < \dots < i_p \leq n,$$

are, as we know, linearly independent. Therefore, if  $A \wedge \mathbf{x}_s = 0$ , then all the coefficients  $A^{i_1 \dots i_p}$  in formula (3) are zero.

This proves that when conditions (1) hold only those coefficients  $A^{i_1 \dots i_p}$  in the expansion (2) may be nonzero for which there is every index  $s = 1, \dots, r$  among the indices  $i_1 < \dots < i_p$ , i.e. such that  $i_1 = 1, \dots, i_r = r$ . Therefore

$$A = B \wedge \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_r,$$

where

$$B = (-1)^{r(p-r)} p! \sum_{r < i_{r+1} < \dots < i_p} A^{1 \dots r i_{r+1} \dots i_p} \mathbf{e}_{i_{r+1}} \wedge \dots \wedge \mathbf{e}_{i_p}. \quad \square$$

**Proposition 2.** *A skew-symmetric functional  $A \in \Lambda^p(\mathcal{V})$  is a multivector if and only if*

$$A \wedge \mathbf{x} = 0$$

*for any associated vector  $\mathbf{x}$ .*

*Proof.* If  $A = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p$ , then since the vectors  $\mathbf{x}_1, \dots, \mathbf{x}_p$  generate a rank space a vector  $\mathbf{x}$  is linearly expressible in terms of them and hence

$$A \wedge \mathbf{x} = \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_p \wedge \mathbf{x} = 0.$$

Conversely, if  $A \wedge \mathbf{x} = 0$  for any associated vector  $\mathbf{x}$ , then  $A \wedge \mathbf{x} = 0$  also for any vector  $\mathbf{x}$  of the rank space. In particular, if  $\mathbf{x}_1, \dots, \mathbf{x}_r$  is a basis of the rank space, then

$$A \wedge \mathbf{x}_1 = 0, \dots, A \wedge \mathbf{x}_r = 0.$$

Consequently, according to Proposition 1 there exists a functional  $B$  of degree  $p - r$  such that

$$A = B \wedge \mathbf{x}_1 \wedge \dots \wedge \mathbf{x}_r.$$

Of course, this is possible only for  $p \geq r$ . Since always  $p \leq r$  (Proposition 2 of Lecture 9), this proves that  $p = r$ , i.e. (Proposition 3 of Lecture 9) that the functional  $A$  is a multivector.  $\square$

By virtue of skew-symmetry any vector associated with a functional  $A$  is given (as an element of a space  $(\mathcal{V}')'$ ) by the formula

$$\mathbf{x}: \xi \mapsto A(\xi, \beta^2, \dots, \beta^p),$$

where  $\beta^2, \dots, \beta^p$  are some fixed covectors, and therefore have the coordinates

$$x^i = A^{ij_2 \dots j_p} \beta_{j_2}^2 \dots \beta_{j_p}^p.$$

Hence the coordinates (coefficients) of the functional  $A \wedge \mathbf{x} = \text{Alt}(A \otimes \mathbf{x})$  are expressed by the formula

$$\begin{aligned} (A \wedge \mathbf{x})^{i_1 i_2 \dots i_{p+1}} &= A^{[i_1 \dots i_p} x^{i_{p+1}]} = \\ &= A^{[i_1 \dots i_p} A^{i_{p+1}] j_2 \dots j_p} \beta_{j_2}^2 \dots \beta_{j_p}^p. \end{aligned}$$

These expressions are equal to zero for all  $\beta_{j_2}^2, \dots, \beta_{j_p}^p$  if and only if

$$A^{[i_1 \dots i_p} A^{i_{p+1}] j_2 \dots j_p} = 0$$

for all  $i_1, \dots, i_p, i_{p+1}, j_2, \dots, j_p$  (if for example  $A^{[i_1 \dots i_p} A^{i_{p+1}] j_2^0 \dots j_p^0} \neq 0$ , then  $(A \wedge \mathbf{x})^{i_1 \dots i_p i_{p+1}} \neq 0$  for

$\beta_{j_2}^2 = \delta_{j_2}^{j_2^0}, \dots, \beta_{j_p}^p = \delta_{j_p}^{j_p^0}$ ). Denoting for the sake of symmetry the index  $i_{p+1}$  by  $j_1$  we see that we have proved the following proposition.

**Proposition 3.** *A skew-symmetric functional  $A \in \Lambda^p(\mathcal{V})$  is a multivector if and only if its coefficients satisfy the relations*

$$(4) \quad A^{[i_1 \dots i_p} A^{j_1] j_2 \dots j_p} = 0$$

for all  $i_1, \dots, i_p, j_1, j_2, \dots, j_p$ .  $\square$

Relations (4) are known as *Plücker relations*.

**Example.** For  $p = 2$  relations (4) have the form

$$A^{[i_1 i_2} A^{j_1] j_2} = 0,$$

i.e. the form

$$A^{i_1 i_2} A^{j_1 j_2} + A^{i_2 j_1} A^{i_1 j_2} + A^{j_1 i_1} A^{i_2 j_2} - A^{i_2 i_1} A^{j_1 j_2} - A^{j_1 i_2} A^{i_1 j_2} - A^{i_1 j_1} A^{i_2 j_2} = 0.$$

By virtue of skew-symmetry the first term is equal to the forth, the second to the fifth and the third to the sixth. Therefore, reducing similar terms and cancelling by 2 we obtain the relation

$$(5) \quad A^{i_1 i_2} A^{j_1 j_2} + A^{i_2 j_1} A^{i_1 j_2} + A^{j_1 i_1} A^{i_2 j_2} = 0.$$

If  $i_1 = i_2$ , then the first term is equal to zero and the other two have different signs. In this case therefore relations (5) hold automatically (by skew-symmetry). The situation is similar when any two of the indices  $i_1, i_2, j_1, j_2$  are equal. Relations (5) are therefore essential if and only if all these four indices are distinct.

Since for  $n = 3$  this is impossible, it follows that all Plücker relations are trivial for  $n = 3$  (and  $p = 2$ ), i.e. *in a three-dimensional space any skew-symmetric functional is a bivector*. This explains why in [1] we managed to convert a set of bivectors into a vector space for  $n = 3$ .

For  $n = 4$  there is only one nontrivial Plücker relation:

$$A^{12} A^{34} + A^{23} A^{14} + A^{13} A^{42} = 0.$$

In this case therefore bivectors do not constitute a vector space (a sum of two bivectors is not in general a bivector).

It can be shown in a similar manner for any  $n$  that if  $p = n - 1$  all Plücker relations are trivial, i.e. that *any skew-symmetric functional of degree  $n - 1$  is an  $n - 1$ -vector*. The essential coefficients  $A^{i_1 \dots i_{n-1}}$ ,  $1 \leq i_1 < \dots < i_{n-1} \leq n$ , of the functional obviously have the form  $A^1 \dots \overset{\wedge}{i} \dots n$  where the sign  $\wedge$  over the index means that that index must be dropped. It is convenient to designate the indicated coefficient by the symbol  $(-1)^i B_i$ .

**Remark.** Since numbers  $B_i$  can be interpreted as the coordinates of some covector, we see that there is a bijection between  $n - 1$ -vectors and covectors (i.e. 1-covectors). It turns out that for any  $p$  there is a similar correspondence between  $p$ -vectors and  $n - p$ -covectors. It depends in general on the choice of basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ , but this dependence is rather weak. That is, this correspondence turns out to be the same for all unimodularly equivalent bases, i.e. those that determine the same  $n$ -vector:

$$E_0 = \mathbf{e}_1 \wedge \dots \wedge \mathbf{e}_n.$$

In this correspondence, to every  $n - p$ -covector  $B = \beta^1 \wedge \dots \wedge \beta^{n-p}$  there corresponds a  $p$ -vector  $A$  defined as a skew-symmetric functional by the formula

$$A(\xi^1, \dots, \xi^p) = E_0(\beta^1, \dots, \beta^{n-p}, \xi^1, \dots, \xi^p).$$

Irrespective of the  $n$ -vector  $E_0$  this correspondence is determined only up to proportionality, i.e. between the classes of proportional  $p$ -vectors and  $n - p$ -covectors. Identifying these classes with subspaces of spaces  $\mathcal{V}$  and  $\mathcal{V}'$  (see the corollary of Proposition 4 in the preceding lecture) leads to a correspondence which associates with each subspace  $\mathcal{P} \subset \mathcal{V}$  its annulet  $\mathcal{P}^\circ$ .

In a Euclidean space, as will be shown in due course, it is possible to identify covectors with vectors and hence  $n - p$ -covectors with  $n - p$ -vectors. It follows that there is a bijection between  $p$ -vectors and  $n - p$ -vectors in an oriented Euclidean space. For  $n = 3$  and  $p = 2$  this correspondence coincides with that introduced by Definition 4 of Lecture 15 in [1].

Unfortunately, we have no possibility to go deeper into these most interesting questions.

It goes without saying that for  $p = n$  Plücker relations are also trivial. This, however, follows also directly from the fact that according to Theorem 1 of Lecture 9 (or, more precisely, its “vector” analogue) the space  $A^n(\mathcal{V})$  is one-dimensional and is generated by any  $n$ -vector  $e_1 \wedge \dots \wedge e_n \neq 0$ . Every skew-symmetric functional of degree  $n$  therefore is an  $n$ -vector of the form  $ae_1 \wedge \dots \wedge e_n$ .

Thus,

$$A^{n-1}(\mathcal{V}) = \Lambda^{n-1}(\mathcal{V}) \text{ and } A^n(\mathcal{V}) = \Lambda^n(\mathcal{V}).$$

Besides, of course,

$$A^0(\mathcal{V}) = \Lambda^0(\mathcal{V}) = \mathbb{K} \text{ and } A^1(\mathcal{V}) = \Lambda^1(\mathcal{V}) = \mathcal{V}.$$

We now apply the results obtained to the geometry of space.

**Definition 1.** A nonzero  $p$ -vector  $A$  is said to be a *direction  $p$ -vector* of a  $p$ -dimensional subspace  $\mathcal{P} \subset \mathcal{V}$  if  $\mathcal{P}$  is its rank subspace. Vectors  $x \in \mathcal{P}$  are also said to be *parallel* to the  $p$ -vector  $A$  (the notation is  $x \parallel A$ ).

According to the corollary of Proposition 4 of the preceding lecture every direction  $p$ -vector is an external product  $x_1 \wedge \dots \wedge x_p$  of vectors of some basis of the subspace  $\mathcal{P}$  and is therefore, up to proportionality, uniquely determined by the subspace  $\mathcal{P}$  (and of course uniquely determines it).

**Definition 2.** The coordinates  $A^{i_1 \dots i_p}$  of an arbitrary direction  $p$ -vector in a  $p$ -dimensional subspace  $\mathcal{P} \subset \mathcal{V}$  are called the *Plücker coordinates* of the subspace. They are determined (for a fixed basis of the space  $\mathcal{V}$ ) up to proportionality, i.e. are homogeneous coordinates.

In terms of an arbitrary basis  $x_1, \dots, x_p$  of the subspace  $\mathcal{P}$  its Plücker coordinates are expressed by the formulas

$$\rho A^{i_1 \dots i_p} = x_1^{i_1} \dots x_p^{i_p}$$

or equivalently by

$$\rho A^{i_1 \dots i_p} = \begin{vmatrix} x_1^{i_1} & \dots & x_1^{i_p} \\ \vdots & \ddots & \vdots \\ x_p^{i_1} & \dots & x_p^{i_p} \end{vmatrix},$$

where  $\rho$  is an arbitrary factor of proportionality.



For a set of numbers  $A^{i_1 \dots i_p}$  to be a set of Plücker coordinates of some  $p$ -dimensional subspace  $\mathcal{F}$  it is necessary and sufficient that

(i) the numbers  $A^{i_1 \dots i_p}$  should be skew-symmetrically dependent on the indices, i.e. that for any permutation  $\sigma \in S_p$  there should be an equation

$$A^{i_{\sigma(1)} \dots i_{\sigma(p)}} = \varepsilon_{\sigma} A^{i_1 \dots i_p};$$

(ii) for any indices  $i_1, \dots, i_p, j_1, \dots, j_p$  the Plücker relation

$$A^{[i_1 \dots i_p} A^{j_1] j_2 \dots j_p} = 0$$

should hold.

This assertion is but an obvious restatement of the results we already know.

Let  $\mathcal{A}$  be an  $n$ -dimensional affine space (see Lecture 5 in [1]) and let  $\mathcal{V}$  be an associated vector space. In complete analogy with the definitions of a straight line and a plane (see Lectures 5 and 6 in [1]) we introduce the following definition.

**Definition 3.** For any point  $M_0 \in \mathcal{A}$  and an arbitrary nonzero  $p$ -vector  $A \in \mathbf{A}^p(\mathcal{V})$  the set of all points  $M \in \mathcal{A}$  for which  $\overrightarrow{M_0 M} \parallel A$  is called a  $p$ -dimensional plane passing through the point  $M_0$  and parallel to the  $p$ -vector  $A$ . The  $p$ -vector is also called a direction  $p$ -vector of the plane.

When  $p = 1$  the plane is called a *straight line* (cf. Definition 7 of Lecture 5 in [1]), and when  $p = n - 1$  it is called a *hyperplane*.

When  $p = 0$  planes are points in a space  $\mathcal{A}$ .

If  $A = \mathbf{a}_1 \wedge \dots \wedge \mathbf{a}_p$  and a point  $O$  is chosen in the space  $\mathcal{A}$ , then the condition  $\overrightarrow{OM} \parallel A$  is equivalent to the equation

$$(6) \quad \mathbf{x} = \mathbf{x}_0 + t^1 \mathbf{a}_1 + \dots + t^p \mathbf{a}_p,$$

where  $\mathbf{x}_0 = \overrightarrow{OM_0}$  and  $\mathbf{x} = \overrightarrow{OM}$  are radius vectors of the points  $M_0$  and  $M$  and  $t^1, \dots, t^p$  are arbitrary numbers. Equation (6) is called the *parametric vector equation* of a plane.





All this means that the “geometric” theory of planes in an  $n$ -dimensional affine space is completely equivalent to the “algebraic” theory of systems of nonhomogeneous linear equations in  $n$  unknowns. Both theories speak of the same things, but in different languages. It is necessary to learn to translate without difficulty from one language to the other.

**Example.** The fact that a system of equations (9) has a unique solution means that the corresponding plane has dimension 0 and is a point in the space  $\mathcal{A}$ . The subspace  $\mathcal{P} \subset \mathcal{V}$  corresponding to it consists of only the zero vector  $\mathbf{0}$  in this case. Consequently the system of homogeneous equations

$$(10) \quad \begin{array}{l} \xi_1^1 x^1 + \dots + \xi_n^1 x^n = 0, \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \xi_1^m x^1 + \dots + \xi_n^m x^n = 0 \end{array}$$

has only one trivial (zero) solution  $(0, \dots, 0)$ . Conversely, if system (10) has only a trivial solution, then the subspace  $\mathcal{P}$  it defines consists of only the vector  $0$ . Every coset  $\mathbf{x} + \mathcal{P}$  therefore consists of only the vector  $\mathbf{x}$  and hence equations (9) have a unique solution. Thus we see that the (compatible) system (9) of nonhomogeneous linear equations has a unique solution if and only if system (10) of homogeneous linear equations has only a trivial solution.

The geometrical fact, equivalent to this algebraic statement, is simply that if  $\mathcal{P} = 0$ , then  $\mathbf{x}_0 + \mathcal{P} = \mathbf{x}_0$  for any  $\mathbf{x}_0 \in \mathcal{V}$ , and conversely.

Similarly a vector  $\mathbf{x}$  belongs by definition to the coset  $\mathbf{x}_0 + \mathcal{P}$  if and only if it is of the form  $\mathbf{x}_0 + \mathbf{a}$ , where  $\mathbf{a} \in \mathcal{P}$ . In “algebraic terms” this means, firstly, that the sum of some fixed solution of system (9) and an arbitrary solution of system (10) is a solution of system (9) and, secondly, that any solution of system (9) can be obtained in this way.

The various situations in relative positions of planes in the space  $\mathcal{A}$  can be algebraically characterized say by the conditions on the ranks of some matrices and their submatrices. We shall not go into this since it is sufficiently dull and at the same time extremely awkward.

The awkwardness of the theory of planes in an affine space is due (at least in part) to the existence of parallel planes. It is therefore natural that in a *projective space* this theory becomes somewhat easier (although remaining sufficiently complicated).

A general definition of an  $n$ -dimensional projective space over an arbitrary field  $\mathbb{K}$  was given in Lecture 26 of [1]. According to the definition one of the models of the space is a set of all one-dimensional subspaces of an  $n + 1$ -dimensional vector space  $\mathbb{K}^{n+1}$ . Instead of  $\mathbb{K}^{n+1}$  we can of course take any  $n + 1$ -dimensional vector space  $\mathcal{V}^{n+1}$  and hence any  $n + 1$ -dimensional affine space  $\mathcal{A} = \mathcal{A}^{n+1}$  with a point  $O$  marked in it. In the last variant the points of the resulting model of a projective space are straight lines of the space  $\mathcal{A}$  passing through the point  $O$ , i.e. we obtain the “bundle” model we already know for the case  $n = 2$  (see Lecture 25 in [1]).

For definiteness we shall consider the model  $\mathbb{P}^n(\mathcal{V})$  whose points are one-dimensional subspaces of an  $n + 1$ -dimensional vector space  $\mathcal{V}$ .

**Definition 5.** A *plane of dimension  $r$*  in a projective space  $\mathbb{P}^n(\mathcal{V})$  is a set of all points of the space that are one-dimensional subspaces of some  $r + 1$ -dimensional subspace  $\mathcal{R} \subset \mathcal{V}$ .

Thus every  $r$ -dimensional plane is by definition an  $r$ -dimensional projective space  $\mathbb{P}^r(\mathcal{R})$ .

Allowing certain inaccuracy (but attaining in return brevity of expression) one ordinarily says that the planes of the space  $\mathbb{P}^n(\mathcal{V})$  are the subspaces of the space  $\mathcal{V}$  (of a dimension higher by unity). Thus, for example, already in Lecture 25 of [1] straight lines of the model  $\mathcal{P}_O$  we identified with planes in an affine space  $\mathcal{A}$  passing through a point  $O$  (i.e. with two-dimensional subspaces of an associated vector space).

As was explained at length (for the case  $n = 2$ ) in Lecture 25 of [1], the *projective coordinates*  $x^0: x^1: \dots: x^n$  of points in the space  $\mathbb{P}^n(\mathcal{V})$  are given by an arbitrary basis  $\mathbf{e}_0, \mathbf{e}_1, \dots, \mathbf{e}_n$  of the space  $\mathcal{V}$ . For every point  $M$  in  $\mathbb{P}^n(\mathcal{V})$  they represent in this basis the coordinates of an arbitrary vector  $\mathbf{x} \in \mathcal{V}$  generating the point as a one-dimensional subspace of the space  $\mathcal{V}$ . This means that the coordinates  $x^0: x^1: \dots$

$\dots :x^n$  are the Plücker coordinates of that one-dimensional subspace.

More generally, we can define (relative to a given projective coordinate system) the *projective coordinates of an arbitrary plane*  $\mathbb{P}^r(\mathcal{R}) \subset \mathbb{P}^n(\mathcal{V})$  as the Plücker coordinates of the subspace  $\mathcal{R}$ . These coordinates are thus of the form  $p^{i_0 i_1 \dots i_r}$ , where  $i_0, i_1, \dots, i_r = 0, 1, \dots, n$ , and are subject to the following conditions:

(i) for any permutation  $\sigma \in S_{r+1}$  we have

$$p^{i_{\sigma(0)} \dots i_{\sigma(r)}} = \varepsilon_{\sigma} p^{i_0 \dots i_r}$$

(it is assumed that  $\sigma$  acts on the numbers  $0, 1, \dots, r$ ); it follows from this condition that there are only  $\binom{n+1}{r+1}$  essential coordinates among the coordinates  $p^{i_0 \dots i_r}$ ; they are, for example, the coordinates

$$p^{i_0 i_1 \dots i_r} \text{ when } i_0 < i_1 < \dots < i_r;$$

(ii) for any indices  $i_0, i_1, \dots, i_r, j_0, j_1, \dots, j_r$  the Plücker relation

$$(11) \quad p^{[i_0 i_1 \dots i_r} p^{j_0] j_1 \dots j_r} = 0$$

holds.

It can be shown that there are exactly

$$(12)' \quad N_{n,r} = \binom{n+1}{r+1} - (r+1)(n-r) - 1$$

independent relations (11).

A straightforward proof of this fact calls for rather sophisticated combinatorics. Next semester we shall develop a general technique for computing such constants with the aid of which the number (12) can be trivially obtained.

It is possible to develop a geometry whose basic elements ("points") are  $r$ -dimensional planes in an  $n$ -dimensional projective space (or equivalently  $r+1$ -dimensional subspaces of an  $n+1$ -dimensional vector space). Analytically this can be done using the coordinates  $p^{i_0 i_1 \dots i_r}$ , i.e. in other words by means of identifying  $r$ -dimensional planes of an

$n$ -dimensional projective space with the points of an  $N$ -dimensional projective space, where  $N = \binom{n+1}{r+1} - 1$ , having the coordinates  $p^{i_0 i_1 \dots i_r}$ ,  $i_0 < i_1 < \dots < i_r$ .

Suppose, for example,  $r = n - 1$  (the case of hyperplanes). Then, as noted above, essential coordinates are the  $n + 1$  coordinates

$$p^{0 \dots \hat{i} \dots n} = (-1)^i q_i, \quad i = 0, 1, \dots, n.$$

On the other hand, hyperplanes are obviously given by a single linear equation for the coordinates  $x^0: x^1: \dots: x^n$ , the numbers  $q_0, q_1, \dots, q_n$ , as can be verified without difficulty (do it!), being exactly the coefficients of the equation. In the first semester (see Lectures 24 and 26 of [1]) we took as the coordinates for straight lines in the plane (the case  $n = 2, r = 1$ ) and planes in space (the case  $n = 3, r = 2$ ) the coefficients of their equations. We thus see that the Plücker coordinates  $p^{i_0 i_1 \dots i_n}$  are a direct generalization of these coordinates.

The fact that for  $r = n - 1$  the coordinates  $p^{i_0 \dots i_n}$  obey no nontrivial Plücker relations (11) means that in representing hyperplanes by points of a  $\left(\binom{n+1}{n} - 1\right)$ -dimensional projective space we obtain the whole of the space. Since  $\binom{n+1}{n} - 1 = n$ , it follows that the geometry of hyperplanes is equivalent to that of points and in any case is not more complicated than the latter.

The situation is different already for straight lines in a three-dimensional space (the case  $n = 3$  and  $r = 1$ ). Here we have  $\binom{4}{2} = 6$  essential coordinates  $p^{i_0 i_1}$ ,  $0 \leq i_0 < i_1 \leq 3$ , which thus determine (in view of homogeneity) a point in a five-dimensional space  $\mathbb{K}P^5$ . Besides, these six coordinates must satisfy one more relation

$$p^{01}p^{23} + p^{12}p^{03} + p^{02}p^{31} = 0$$

(see above; the indices are decreased by 1 because now they take values from 0 to 3), which defines in  $\mathbb{K}P^5$  a "second-degree hypersurface". Thus we see that the geometry of

straight lines in space is equivalent to that of points of some “curved” hypersurface in a five-dimensional space. It is no wonder therefore that the geometry of straight lines in space is much more complicated than, say, the geometry of planes. It is for this reason that we actually entirely ignored this geometry in the first semester.

Still more complicated of course is the situation for any  $r$  and  $n$ . The variety of  $r$ -dimensional planes of an  $n$ -dimensional space is represented by points of a  $\left(\binom{n+1}{r+1} - 1\right)$ -dimensional space lying in the intersection  $N_{n,r}$  of second-degree hypersurfaces. This intersection is called the *Grassman manifold* and has been intensively studied for many years. We have not yet got a comprehensive knowledge of its geometry, however.



# Lecture II

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Symmetric and skew-symmetric bilinear functionals • A matrix of symmetric bilinear functionals • The rank of a bilinear functional • Quadratic functionals and quadratic forms • Lagrange theorem

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By analogy with skew-symmetric functionals *symmetric multilinear functionals* are defined to be functionals  $B$  in  $\mathcal{T}_p(\mathcal{V})$  (or in  $\mathcal{T}^p(\mathcal{V})$ ) such that

$$\sigma B = B$$

for any permutation  $\sigma \in S_p$ . The theory of such functionals, however, turns out to be very complicated and up to now very little is known about them in the general case. The only exception is the case  $p = 2$ , i.e. the case of bilinear functionals. We shall now deal with these functionals. For definiteness we shall consider functionals of vectors (i.e. in  $\mathcal{T}_2(\mathcal{V})$ ).

Since the group  $S_2$  consists of two elements only, an identity permutation and a transposition  $(1\ 2)$ , a *bilinear functional*  $B$  is

$$\left. \begin{array}{l} \text{symmetric} \\ \text{skew-symmetric} \end{array} \right\} \text{ when } \left\{ \begin{array}{l} B(\mathbf{x}, \mathbf{y}) = B(\mathbf{y}, \mathbf{x}), \\ B(\mathbf{x}, \mathbf{y}) = -B(\mathbf{y}, \mathbf{x}) \end{array} \right.$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ .

Just as skew-symmetric functionals constitute a subspace  $\Lambda_2(\mathcal{V})$  of a space  $\mathcal{T}_2(\mathcal{V})$  so the set  $\mathcal{S}_2(\mathcal{V})$  of all symmetric bilinear functionals is a subspace of the space  $\mathcal{T}_2(\mathcal{V})$ .

If the characteristic of the ground field  $\mathbb{K}$  is two, then  $\Lambda_2(\mathcal{V}) = \mathcal{S}_2(\mathcal{V})$ . But if the characteristic of the field  $\mathbb{K}$  is

other than two, then the subspaces  $\Lambda_2(\mathcal{V})$  and  $S_2(\mathcal{V})$  are obviously disjoint:

$$\Lambda_2(\mathcal{V}) \cap S_2(\mathcal{V}) = 0.$$

In what follows we always assume this to hold.

**Proposition 1.** *A space  $T_2(\mathcal{V})$  is a direct sum of spaces  $\Lambda_2(\mathcal{V})$  and  $S_2(\mathcal{V})$ , i.e. any bilinear functional  $B$  can be uniquely represented as a sum of symmetric functional  $B_{\text{symm}}$  and a skew-symmetric functional  $B_{\text{skew}}$ :*

$$(1) \quad B = B_{\text{symm}} + B_{\text{skew}}.$$

*Proof.* The uniqueness of expansion (1) is ensured by the disjointness of the spaces  $\Lambda_2(\mathcal{V})$  and  $S_2(\mathcal{V})$  and in order to find at least one such expansion it is sufficient to put

$$B_{\text{symm}} = \frac{B + \sigma B}{2}, \quad B_{\text{skew}} = \frac{B - \sigma B}{2},$$

where  $\sigma$  is a transposition (1 2).  $\square$

Note that  $B_{\text{skew}}$  is none other but  $\text{Alt } B$ .

As we know (see Lecture 5), in a given basis  $e_1, \dots, e_n$  of a space  $\mathcal{V}$  a functional  $B$  is uniquely defined by its matrix

$$(2) \quad \begin{pmatrix} b_{11} & \dots & b_{1n} \\ b_{n1} & \dots & b_{nn} \end{pmatrix}$$

the elements  $b_{ij}$  of which are given by the formula

$$b_{ij} = B(e_i, e_j), \quad i, j = 1, \dots, n.$$

The value  $B(x, y)$  of the functional  $B$  on arbitrary vectors  $x, y \in \mathcal{V}$  is a bilinear form of their coordinates:

$$B(x, y) = b_{ij} x^i y^j$$

with coefficients  $b_{ij}$ . It readily follows that a functional  $B$  is symmetric if and only if so is its matrix, i.e.

$$b_{ij} = b_{ji}$$

for any  $i, j = 1, \dots, n$ ,

Indeed, if a functional  $B$  is symmetric, then in particular  $B(\mathbf{e}_i, \mathbf{e}_j) = B(\mathbf{e}_j, \mathbf{e}_i)$ , i.e.  $b_{ij} = b_{ji}$ . Conversely, if  $b_{ij} = b_{ji}$ , then for any vectors  $\mathbf{x}, \mathbf{y}$  we have

$$B(\mathbf{y}, \mathbf{x}) = b_{ij}y^i x^j = b_{ji}y^j x^i = b_{ij}x^i y^j = B(\mathbf{x}, \mathbf{y})$$

and therefore the functional  $B$  is symmetric.

For bilinear functionals, just as for any multilinear functionals, the concept of rank is defined, i.e. (see Lecture 7) that of the smallest number of covectors in terms of which a functional is expressible in tensor form. On the other hand, especially for the bilinear functional it is possible to speak of the rank of its matrix (2). One would like to think that the two concepts of rank coincide. However, in general this is not true.

**Example.** Let  $n = 2$  and  $B = \mathbf{e}^1 \otimes \mathbf{e}^2$ . The matrix of this functional has the form

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and hence its rank equals unity. On the other hand, in tensor form the functional  $B$  is expressible in terms of two covectors and cannot be expressed in terms of one covector (if only because any bilinear functional of rank 1 is symmetric).

**Definition 1.** The rank of matrix (2) is called the *matrix rank* of a bilinear functional  $B$ .

As we know (see Lecture 5), when changing to another basis the matrix of the functional  $B$  is multiplied on the left and right by matrices  $C^\top$  and  $C$ , where  $C$  is the transition matrix. Since when multiplied by a nonsingular matrix the rank of the matrix remains unaltered (Proposition 2 of Lecture 2) this shows that Definition 1 is correct.

**Proposition 2.** The matrix rank  $r_{\text{mat}}$  of a bilinear functional  $B$  does not exceed its rank  $r$ :

$$r_{\text{mat}} \leq r.$$

*Proof.* Let  $\xi^1, \dots, \xi^r$  be a basis of the rank space of the functional  $B$ . Then

$$B = \sum_{i,j=1}^r b_{ij} \xi^i \otimes \xi^j,$$

where  $b_{ij}$ ,  $i, j = 1, \dots, r$  are some numbers. Now if we extend the basis  $\xi^1, \dots, \xi^r$  to a basis

$$\mathbf{e}^1 = \xi^1, \dots, \mathbf{e}^r = \xi^r, \mathbf{e}^{r+1}, \dots, \mathbf{e}^n$$

of a space  $\mathcal{V}'$ , then in the corresponding basis  $\mathbf{e}^i \otimes \mathbf{e}^j$ ,  $i, j = 1, \dots, n$ , of a space  $\mathbb{T}_2(\mathcal{V})$  (see Proposition 1 of Lecture 5) the formula

$$B = \sum_{i,j=1}^r b_{ij} \mathbf{e}^i \otimes \mathbf{e}^j$$

will hold. This means that in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  the matrix of the functional has the form

$$\begin{pmatrix} b_{11} & \dots & b_{1r} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{r1} & \dots & b_{rr} & 0 & \dots & 0 \\ 0 & \dots & \dots & \dots & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Therefore its rank does not exceed  $r$ .  $\square$

For symmetric bilinear functionals the situation is much more satisfactory.

**Proposition 3.** *The matrix rank of a symmetric bilinear functional  $B$  coincides with its rank:*

$$r_{\text{mat}} = r.$$

*Proof.* By virtue of symmetry any associated covector has the form

$$(3) \quad \mathbf{x} \mapsto B(\mathbf{x}, \mathbf{a}),$$

where  $\mathbf{a}$  is an arbitrary vector. Since all such covectors are obviously linearly expressible in terms of covectors

$$(4) \quad \mathbf{x} \mapsto B(\mathbf{x}, \mathbf{e}_i), \quad i = 1, \dots, n,$$

this proves that the rank space  $\mathcal{R}$  of the functional  $B$  is generated by covectors (4). Therefore the rank  $r = \dim \mathcal{R}$  equals the rank of the family of covectors (4). But it is clear that the coordinates (in the basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$ ) of covectors (4)

are the columns of matrix (2). Hence the rank  $r$  equals the rank of this matrix.  $\square$

Note that this proof obviously remains valid for skew-symmetric bilinear functionals as well, so that  $r_{\text{mat}} = r$  for them too.

Besides, in both cases covectors of the form (3) obviously form a subspace. Hence for (skew) symmetric bilinear functionals the rank subspace consists of associated covectors (not only is merely generated by them).

**Definition 2.** A functional  $Q: x \rightarrow Q(x) \in \mathbb{K}$  is said to be *quadratic* if there exists a bilinear functional  $B$  such that

$$(5) \quad Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$$

for any vector  $\mathbf{x} \in \mathcal{V}$ .

Expanding  $B$  by formula (1) and taking into account the fact that  $B_{\text{skew}}(\mathbf{x}, \mathbf{x}) = 0$  we find that the functional  $B$  in formula (5) may be assumed to be symmetric without loss of generality.

It is easy to see that then the functional  $B$  is uniquely determined by the functional  $Q$ , i.e. in other words the *correspondence*

$$B \mapsto Q$$

is a *bijection* between a vector space  $S_2(\mathcal{V})$  of symmetric bilinear functionals and the set of all quadratic functionals in  $\mathcal{V}$  (we assume as before that the characteristic of the ground field  $\mathbb{K}$  is other than two).

It is indeed clear that if  $Q(\mathbf{x}) = B(\mathbf{x}, \mathbf{x})$  and the functional  $B$  is symmetric, then

$$\frac{Q(\mathbf{x} + \mathbf{y}) - Q(\mathbf{x}) - Q(\mathbf{y})}{2} = B(\mathbf{x}, \mathbf{y})$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ .  $\square$

It makes no difference whatsoever in principle therefore whether one considers symmetric bilinear or quadratic functionals, for any statement about quadratic functionals can be reformulated as a statement about symmetric bilinear functionals and vice versa. We choose quadratic functionals as basic leaving it to the reader to reformulate the statements about them in terms of symmetric bilinear functionals.











formation we obtain a basis  $e_1'', e_2'', \dots, e_n''$  the first two vectors of which  $e_1''$  and  $e_2''$  are  $Q$ -orthogonal to each other and to the other vectors, and so on.

Where this construction continues indefinitely, i.e. every time (until we exhaust the basis or obtain a zero functional) the basic transformation is applicable, Theorem 1 thus turns out to be proved. This case is said to be *regular*.

But if at some stage the basic transformation (9) turns out to be inapplicable, then one should make auxiliary transformations which result in a basis to which transformation (9) is now applicable.

*First auxiliary transformation.* It is applied when  $q_{11} = 0$  but there is an index  $i_0$  such that  $q_{i_0 i_0} \neq 0$ . It consists in permuting the  $i_0$ th vector of the basis to the first place

$$\begin{aligned} e_1' &= e_{i_0}, \quad e_{i_0}' = e_1, \\ e_i' &= e_i \quad \text{if } i \neq 1, i_0. \end{aligned}$$

It is obvious that in the new basis  $q'_{11} \neq 0$ .

*Second auxiliary transformation.* It is applied when  $q_{ii} = 0$  for all  $i = 1, \dots, n$  but the functional  $Q$  is not zero and therefore there exist indices  $i_0$  and  $j_0$  such that  $q_{i_0 i_0} \neq 0$ . If, for example,  $q_{12} \neq 0$  (this assumption does not lead to any loss of generality, of course), then the transformation considered is given by the formulas

$$\begin{aligned} e_1' &= e_1 + e_2, \\ e_i' &= e_i, \quad \text{if } i \geq 2. \end{aligned}$$

Then

$$q'_{11} = Q(e_1') = Q(e_1 + e_2, e_1 + e_2) = 2q_{12} \neq 0$$

and it is possible to apply the basic transformation.

None of the transformations is applicable if and only if all the coefficients  $q_{ij}$  are zero, i.e. if  $Q = 0$ . But in this case any basis is obviously  $Q$ -orthogonal and therefore one need not do anything with it.

Consequently, applying our transformations in the necessary succession we sooner or later obtain a  $Q$ -orthogonal basis.  $\square$

In a  $Q$ -orthogonal basis a matrix of the form  $Q$  is obviously diagonal, i.e. has the form

$$(10) \quad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

and hence

$$Q(\mathbf{x}) = \lambda_1 (x^1)^2 + \dots + \lambda_n (x^n)^2$$

for any vector  $\mathbf{x}$ . In terms of quadratic forms therefore the Lagrange theorem asserts that *any quadratic form*  $Q(x^1, \dots, x^n)$  *is equivalent to a form*

$$(11) \quad \lambda_1 (x^1)^2 + \dots + \lambda_n (x^n)^2.$$

The form (11) is said to be of *normal form*. Thus we see that *any quadratic form*  $Q(x^1, \dots, x^n)$  *can be reduced to a normal form (11) by means of a nonsingular linear transformation (7).*

The last statement also known as the *Lagrange theorem* fully relates to algebra and all traces of its geometric origin have disappeared in it. It is therefore applicable to quadratic forms arising in any questions (say in mechanics) that are *a priori* in no way connected with the geometry of quadratic functionals.

In practice, reducing a quadratic form  $Q(x^1, \dots, x^n)$  to normal form should be carried out by successively "selecting squares", i.e. by using the identity

$$Q(x^1, \dots, x^n) = \frac{1}{q_{11}} (q_{11} x^1 + \dots + q_{1n} x^n)^2 + Q'(x^2, \dots, x^n),$$

where the form  $Q'$ , as can be easily seen, has already no variable  $x^1$ . This identity corresponds to the basic Lagrange transformation. In the irregular case one has in addition to renumber the variables and use transformations of the form

$$\begin{aligned} y_1 &= x_1 - x_2, \\ y_i &= x_i, \text{ if } i \geq 2. \end{aligned}$$

Some of the coefficients  $\lambda_1, \dots, \lambda_n$  (or all of them) of the form (11) may be zero. It is clear that the number  $r$

of the nonzero coefficients equals the rank of the matrix (6) and hence the rank of the functional  $Q$ . Transposing the elements of the basis, if necessary, we can always see to it that the first coefficients  $\lambda_1, \dots, \lambda_r$  should be nonzero. Since it is not necessary to write the terms with zero coefficients we finally find that the *normal quadratic form of rank  $r$  is the form*

$$\lambda_1 (x^1)^2 + \dots + \lambda_r (x^r)^2, \text{ where } \lambda_1 \neq 0, \dots, \lambda_r \neq 0.$$

# Lecture 12

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*Jacobi theorem • Quadratic forms over the fields of complex and real numbers • The law of inertia • Positively definite quadratic functionals and forms*

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Recall that a quadratic matrix is said to be *triangular* (more precisely, upper-triangular) if all its elements below the principal diagonal are zero. The determinant of such a matrix is obviously equal to the product of the diagonal elements of the matrix. A triangular matrix therefore is non-singular if and only if all its diagonal elements are nonzero. Of particular importance are triangular matrices all diagonal elements of which are equal to unity. We shall call such matrices *unitriangular matrices*.

A direct computation shows that a product of two (uni)-triangular matrices and the inverse of a (uni)triangular matrix are also (uni)triangular matrices.

Since the matrix of the basic transformation in the Lagrange algorithm is a unitriangular matrix

$$\begin{pmatrix} 1 - \frac{q_{12}}{q_{11}} & \dots & -\frac{q_{1n}}{q_{11}} \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{pmatrix},$$

it follows that *in the regular case transition to a  $Q$ -orthogonal basis is effected by a unitriangular matrix.*

Let  $A$  be an arbitrary quadratic matrix of order  $n$  and  $1 \leq k \leq n$ . Eliminating the last  $n - k$  rows and  $n - k$  columns from the matrix  $A$  we obtain a quadratic matrix of order  $k$ .

**Definition 1.** This matrix is called the *principal submatrix of order  $k$*  of the matrix  $A$  and its determinant is called the *principal minor of order  $k$*  of the matrix  $A$ .

Let  $Q$  and  $Q'$  be matrices of a quadratic functional  $Q$  in two bases  $e_1, \dots, e_n$  and  $e'_1, \dots, e'_n$  connected by a unitriangular transition matrix  $C$ . Then the following obvious assertions hold.

(a) The principal submatrix  $C_k$  of order  $k$  of the matrix  $C$  is a transition matrix connecting the bases  $e_1, \dots, e_k$  and  $e'_1, \dots, e'_k$  of a subspace  $\mathcal{P}_k = [e_1, \dots, e_k] = [e'_1, \dots, e'_k]$ .

(b) The restriction  $Q|_{\mathcal{P}_k}$  of the functional  $Q$  to the subspace  $\mathcal{P}_k$  is a quadratic functional whose matrix in the basis  $e_1, \dots, e_k$  is the principal submatrix  $Q_k$  of order  $k$  of the matrix  $Q$ , the matrix in the basis  $e'_1, \dots, e'_k$  being the principal submatrix  $Q'_k$  of order  $k$  of the matrix  $Q'$ .

It follows that

$$Q'_k = C_k^\top Q_k C_k$$

for any  $k = 1, \dots, n$ . Switching to determinants and considering that  $\det C_k^\top = \det C_k = 1$  it follows that

$$(1) \quad \det Q'_k = \det Q_k$$

for any  $k = 1, \dots, n$ .

In particular, if

$$Q' = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

then

$$\det Q_k = \lambda_1 \dots \lambda_k$$

for any  $k = 1, \dots, n$ .

This proves (we pass to the language of quadratic forms) that if for a quadratic form  $Q(x^1, \dots, x^n)$  the regular case holds then the coefficients  $\lambda_1, \dots, \lambda_n$  of its normal form satisfy the relations

$$(2) \quad \lambda_1 \dots \lambda_k = D_k, \quad k = 1, \dots, n,$$

where  $D_k$  are the principal minors of the quadratic form.  $\square$

Note now that carrying out the basic transformation in the Lagrange algorithm we obtain every time a nonzero coefficient  $\lambda$  (for example the very first transformation yields the coefficient  $\lambda_1 = q_{11} \neq 0$ ). In the regular case the process comes to a stop when after some (say the  $r$ th) step we obtain an identical zero (so that all the remaining coefficients  $\lambda_{r+1}, \dots, \lambda_n$  turn out to be zero). It follows from this and relations (2) as well, firstly, that in the regular case

$$(3) \quad D_1 \neq 0, \dots, D_r \neq 0,$$

where  $r$  is the rank of the form (functional) and, secondly, that

$$\lambda_1 = D_1, \lambda_2 = \frac{D_2}{D_1}, \dots, \lambda_r = \frac{D_r}{D_{r-1}}.$$

Conversely, suppose that for the matrix of a quadratic form inequalities (3), where  $r$  is the rank of the matrix, hold. Then, since  $q_{11} = D_1 \neq 0$  the basic transformation of the Lagrange algorithm is applicable to the form. According to formula (1) the principal minors of the matrix  $Q'$  resulting from the transformation will coincide with those of the matrix  $Q$  and therefore this matrix will possess properties (3) as before. But the principal minor  $D_2$  of the matrix  $Q'$  is obviously equal to the product  $q'_{11}q'_{22}$  (where  $q'_{11} = q_{11} = \lambda_1$ ) and hence  $q'_{22} \neq 0$ . Consequently the basic Lagrange transformation is applicable again to the restriction of the functional  $Q$  to the subspace  $[e'_2, \dots, e'_n]$ , etc.

After  $r$  steps we obtain a matrix of the form

$$(4) \quad \begin{pmatrix} \lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \lambda_r \\ 0 & & & & \overline{G} \end{pmatrix}$$

where  $\lambda_1 \neq 0, \dots, \lambda_r \neq 0$  and  $G$  is some matrix. But since the matrices of the functional have the same rank  $r$  in all the bases, the matrix (4) has rank  $r$  too, which is obviously possible if and only if all the elements of the matrix  $G$  are zero.

Thus the matrix (4) is the matrix of a normal quadratic form and since we have obtained it using only the basic

transformations of the Lagrange algorithm it follows that the regular case holds for the original form.

We have thus proved the following theorem.

**Theorem 1 (Jacobi theorem).** *For a quadratic form of rank  $r$  the regular case holds if and only if the principal minors of the form are nonzero:*

$$D_1 \neq 0, \dots, D_r \neq 0.$$

*The Lagrange algorithm reduces such a form to*

$$D_1 (x^1)^2 + \frac{D_2}{D_1} (x^2)^2 + \dots + \frac{D_r}{D_{r-1}} (x^r)^2. \quad \square$$

This theorem is often very helpful.

A further simplification of the normal form

$$(5) \quad \lambda_1 (x^1)^2 + \dots + \lambda_r (x^r)^2$$

of a quadratic form depends on the arithmetic properties of the field  $\mathbb{K}$ . The simplest case arises when  $\mathbb{K} = \mathbb{C}$ . Using in this case a transformation of the form

$$\begin{aligned} y^1 &= \sqrt{\lambda_1} x^1, \\ &\dots \dots \dots \\ y^r &= \sqrt{\lambda_r} x^r, \\ y^{r+1} &= x^{r+1}, \\ &\dots \dots \dots \\ y^n &= x^n \end{aligned}$$

we can reduce the form (5) to the following (we omit the primes in the notation for coordinates)

$$(6) \quad (x^1)^2 + \dots + (x^r)^2.$$

This proves the following proposition.

**Proposition 1.** *Any quadratic form over the field  $\mathbb{C}$  (i.e. with coefficients in  $\mathbb{C}$ ) can be reduced by a linear nonsingular transformation of variables (also with coefficients in  $\mathbb{C}$ ) to the form (6) where  $r$  is the rank of the form.  $\square$*

In other words, any quadratic form  $Q(x^1, \dots, x^n)$  of rank  $r$  over the field  $\mathbb{C}$  is of the form

$$\varphi_1(x)^2 + \dots + \varphi_r(x)^2,$$



where  $\varphi_1(x), \dots, \varphi_r(x)$  are linearly independent linear forms in  $x^1, \dots, x^n$ .

**Corollary (theorem on the classification of quadratic forms over the field  $\mathbb{C}$ ).** *Two quadratic forms over the field  $\mathbb{C}$  are equivalent if and only if their ranks are equal.  $\square$*

Over the field  $\mathbb{R}$  of real numbers we can make the transformation

$$\begin{aligned} y^1 &= \sqrt{|\lambda_1|} x^1, \\ &\dots \dots \dots \\ y^r &= \sqrt{|\lambda_r|} x^r, \\ y^{r+1} &= x^{r+1}, \\ &\dots \dots \dots \\ y^n &= x^n, \end{aligned}$$

reducing the form (5) (possibly after some additional rearrangement of coordinates) to the form (we again omit the primes in the coordinates)

$$(7) \quad (x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^r)^2,$$

where  $r$  is the rank of the form and  $p$  some number (satisfying the inequalities  $0 \leq p \leq r$ ).

This proves the following proposition.

**Proposition 2.** *Any quadratic form over the field  $\mathbb{R}$  can be reduced by a linear nonsingular transformation of its variables to the form (7) where  $r$  is the rank of the form and  $0 \leq p \leq r$ .  $\square$*

In connection with Proposition 2 the question immediately arises as to whether it is possible or not to reduce a given quadratic form to two forms (7) with distinct  $p$ . It turns out that the answer to this question is negative.

**Proposition 3 (the law of inertia of quadratic forms).** *If two forms*

$$(8) \quad (x^1)^2 + \dots + (x^p)^2 - (x^{p+1})^2 - \dots - (x^r)^2$$

and

$$(9) \quad (y^1)^2 + \dots + (y^q)^2 - (y^{q+1})^2 - \dots - (y^r)^2$$

are equivalent (over the field  $\mathbb{R}$ ), then  $p = q$ .

*Proof.* The equivalence of the forms (8) and (9) means that they are expressions in two different bases  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{f}_1, \dots, \mathbf{f}_n$  for the same quadratic functional  $Q$  given in an  $n$ -dimensional vector space  $\mathcal{V}$ . Let  $\mathcal{P}$  be a subspace of the space  $\mathcal{V}$  generated by vectors  $\mathbf{e}_1, \dots, \mathbf{e}_p$  and let  $\mathcal{Q}$  be a subspace of  $\mathcal{V}$  generated by vectors  $\mathbf{f}_{q+1}, \dots, \mathbf{f}_n$ :

$$\mathcal{P} = [\mathbf{e}_1, \dots, \mathbf{e}_p], \quad \mathcal{Q} = [\mathbf{f}_{q+1}, \dots, \mathbf{f}_n].$$

Since the functional  $Q$  is expressed by the form (8) in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ , for any nonzero vector  $\mathbf{x} \in \mathcal{P}$  we have the relation

$$Q(\mathbf{x}) = (x^1)^2 + \dots + (x^p)^2 > 0.$$

Similarly, for any vector  $\mathbf{y} \in \mathcal{Q}$  we have

$$Q(\mathbf{y}) = -(y^{q+1})^2 - \dots - (y^n)^2 \leq 0.$$

Therefore  $\mathcal{P} \cap \mathcal{Q} = 0$ , i.e. the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  are disjoint and hence (Corollary 2 of Theorem 1 in Lecture 1) for their dimensions there holds the inequality

$$\dim \mathcal{P} + \dim \mathcal{Q} \leq n,$$

i.e. the inequality

$$p + (n - q) \leq n$$

equivalent to the inequality

$$p \leq q.$$

Similarly for  $q \leq p$ . Therefore  $p = q$ .  $\square$

Proposition 3 guarantees the correctness of the following definition.

**Definition 2.** The number  $p$  of “positive squares” in the reduced form (7) is called the *positive inertial index* of a given quadratic form (quadratic functional) and the number  $r - p$  of “negative squares” is called the *negative inertial index*.

In addition Proposition 3 immediately yields the following corollary.

**Corollary (theorem of the classification of quadratic forms over the field  $\mathbb{R}$ ).** *Two quadratic forms over the field  $\mathbb{R}$  are equivalent if and only if their ranks and inertial indices coincide.*

Of particular importance in vector spaces over the field  $\mathbb{R}$  are quadratic functionals  $Q$  possessing the property that  $Q(\mathbf{x}) > 0$  when  $\mathbf{x} \neq 0$ . Their importance is due to the fact that the corresponding symmetric bilinear functionals are precisely all possible scalar multiplications in  $\mathcal{V}$  (see Definition 2 of Lecture 13 in [1]).

**Definition 3.** A quadratic functional  $Q$  in a real vector space  $\mathcal{V}$  is said to be *positive definite* if  $Q(\mathbf{x}) > 0$  for any vector  $\mathbf{x} \neq 0$ .

A quadratic form  $Q(x^1, \dots, x^n)$  is said to be *positive definite* if it is an expression for a positive definite functional in some basis, i.e. in other words if  $Q(x^1, \dots, x^n) > 0$  when  $(x^1, \dots, x^n) \neq (0, \dots, 0)$ .

A matrix  $Q$  is said to be *positive definite* if it is the matrix of a positive definite quadratic functional (quadratic form), i.e. in other words is the matrix of the metric coefficients of some basis of a Euclidean space (see Lecture 14 in [1]).

**Proposition 4.** A quadratic functional (quadratic form) is positive definite if and only if its rank and positive inertial index are equal to  $n$ :

$$r = n, \quad p = n.$$

*Proof.* If  $p = r = n$ , then in some basis the functional  $Q$  is expressed by the form

$$(x^1)^2 + \dots + (x^n)^2$$

and hence  $Q(\mathbf{x}) = 0$  if and only if  $x^1 = 0, \dots, x^n = 0$ , i.e. if  $\mathbf{x} = 0$ .

Conversely, if  $p < n$  or  $r < n$  then in some basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  the functional  $Q$  is expressed by a form

$$Q'(x^1, \dots, x^{n-1}) + \varepsilon (x^n)^2,$$

where  $Q'(x^1, \dots, x^{n-1})$  is a quadratic form in the coordinates  $x^1, \dots, x^{n-1}$  and  $\varepsilon \leq 0$ . Then  $Q(\mathbf{e}_n) = \varepsilon \leq 0$  and hence the functional  $Q$  is not positively definite.  $\square$

This proposition involves a preliminary reduction of the quadratic form to its normal form and hence tends to be useless in practice. Of more interest is the following proposition providing the necessary and sufficient conditions for the positive definiteness of a quadratic form directly from its matrix.

**Proposition 5 (Sylvester's criterion).** *A matrix  $Q$  is positive definite if and only if all of its principal minors are positive:*

$$q_{11} > 0, \begin{vmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{vmatrix} > 0, \begin{vmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{vmatrix} > 0, \dots, \begin{vmatrix} q_{11} & \dots & q_{1n} \\ \cdot & \cdot & \cdot \\ q_{n1} & \dots & q_{nn} \end{vmatrix} > 0.$$

*Proof.* If all principal minors of the matrix  $Q$  are positive (and hence nonzero), then by Theorem 1 for a quadratic form with matrix  $Q$  the regular case holds and the form reduces to the form

$$D_1 (x^1)^2 + \frac{D_2}{D_1} (x^2)^2 + \dots + \frac{D_n}{D_{n-1}} (x^n)^2,$$

where  $D_1 > 0$ ,  $D_2 > 0$ ,  $\dots$ ,  $D_n > 0$ . Thus  $p = r = n$  and therefore the quadratic form (and hence also the matrix) is positive definite.

Conversely, if a form with matrix  $Q$  is positive definite then it can be reduced to a sum of  $n$  squares, i.e. to a form with unit matrix  $E$ . Therefore (cf. Lecture 14 of [1]) the matrix  $Q$  has the form

$$Q = C^\top C,$$

where  $C$  is some nonsingular matrix. Hence

$$\det Q = (\det C)^2 > 0.$$

This proves that the *determinant of a positive definite matrix is positive*.

On the other hand, on setting in a quadratic form  $Q(x^1, \dots, x^n)$  in  $n$  variables the last  $n - k$  variables  $x^{k+1}, \dots, x^n$  equal to zero we obtain a quadratic form

$$Q_k(x^1, \dots, x^k) = Q(x^1, \dots, x^k, 0, \dots, 0)$$

in  $k$  variables  $x^1, \dots, x^k$  for which obviously are true the following assertions.

(a) If a form  $Q(x^1, \dots, x^n)$  is positive definite, so is the form  $Q_k(x^1, \dots, x^k)$ .

(b) A matrix of the form  $Q_k(x^1, \dots, x^k)$  serves as the principal submatrix  $D_k$  of order  $k$  of a matrix of the form  $Q(x^1, \dots, x^n)$ .

Consequently, by virtue of the above remark all principal minors  $D_k$ ,  $k = 1, \dots, n$ , of a positive definite matrix are positive.

Proposition 5 answers in particular the question, put in Lecture 14 of [1], concerning the necessary and sufficient conditions a quadratic matrix must satisfy in order to be the matrix of the coefficients of some basis of a Euclidean space.

# Lecture 13

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*Second degree hypersurfaces in an  $n$ -dimensional projective space • Second degree hypersurfaces in a complex and a real-complex projective space • Second degree hypersurfaces in an  $n$ -dimensional affine space • Second degree hypersurfaces in a complex and a real-complex affine space*

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Let us apply the results obtained on quadratic forms in the preceding lectures to the investigation of second degree hypersurfaces in an  $n$ -dimensional projective space.

**Definition 1** (cf. Definition 2 of Lecture 25 in [1]). A *second degree hypersurface* in an  $n$ -dimensional projective space (over an arbitrary field  $\mathbb{K}$  of characteristic other than two) is a set of points whose projective coordinates  $x_0 : x_1 : \dots : x_n$  satisfy an equation of the form

$$Q(x_0, x_1, \dots, x_n) = 0$$

where  $Q(x_0, x_1, \dots, x_n)$  is some quadratic form in the coordinates  $x_0, x_1, \dots, x_n$ . (Now it is convenient to use subscripts in the coordinates.)

The Lagrange theorem immediately yields the following theorem.

**Theorem 1** (reduction of the equations of second degree hypersurfaces in an  $n$ -dimensional projective space over an arbitrary field  $\mathbb{K}$  to normal form). *For any second degree hypersurface in an  $n$ -dimensional projective space over a field  $\mathbb{K}$  of characteristic other than two there exists a system of projective coordinates  $x_0 : x_1 : \dots : x_n$  in which the equation of the hypersurface has the form*

$$(1) \quad \lambda_0 x_0^2 + \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = 0,$$

where  $0 \leq r \leq n$  and  $\lambda_1 \neq 0, \dots, \lambda_r \neq 0$ .  $\square$

When  $r = n$  the hypersurface (1) is called an *oval second degree hypersurface*.

It is obvious that for any  $k-1$ -dimensional plane  $\Pi_0$  in a projective space and any point  $M \notin \Pi_0$  in it there exists a unique  $k$ -dimensional plane  $M\Pi_0$  containing  $M$  and  $\Pi_0$ .

**Definition 2.** A hypersurface in an  $n$ -dimensional projective space (over an arbitrary field  $\mathbb{K}$ ) is said to be a  *$k$ -fold cylinder* (or a  *$k$ -fold cone*, the concepts of cylinder and cone coinciding in projective space) if there exists a  $k-1$ -dimensional plane  $\Pi_0$  (the *axial plane* of the cylinder) such that for any point  $M$  of the hypersurface not lying in the plane  $\Pi_0$  the plane  $M\Pi_0$  lies entirely in the hypersurface.

Every  $n-k$ -dimensional plane  $\Pi$  having no points in common with the plane  $\Pi_0$  cuts the cylinder in a hypersurface in  $\Pi$  which is called a *base of the cylinder*. A cylinder is also said to be a cylinder *over* its base. It is obvious that every cylinder is a union of all  $k$ -dimensional planes of the form  $M\Pi_0$ , where  $M$  is an arbitrary point in the base of the cylinder. In this sense the geometry of a cylinder is completely reducible to the geometry of its base.

Having all this in mind, consider a hypersurface (1) for  $r < n$ . Let  $\Pi$  be a plane of dimension  $r$  defined by  $n-r$  equations  $x_{r+1} = 0, \dots, x_n = 0$ . In the plane the numbers  $x_0, x_1, \dots, x_r$  are projective coordinates and in these equation (1) defines some oval second degree hypersurface. Also let  $\Pi_0$  be a plane of dimension  $n-r-1$  defined by  $r+1$  equations  $x_0 = 0, x_1 = 0, \dots, x_r = 0$ .

The fact that together with some point  $(x_0^{(0)} : x_1^{(0)} : \dots : x_n^{(0)})$  the hypersurface (1) contains all points of the form  $(x_0^{(0)} : x_1^{(0)} : \dots : x_r^{(0)} : x_{r+1} : \dots : x_n)$ , where  $x_{r+1}, \dots, x_n$  are arbitrary numbers, obviously means that *that hypersurface is an  $n-r$ -fold cylinder* with axial plane  $\Pi_0$ . Serving as the base of the cylinder is the hypersurface defined in the plane  $\Pi$  by (1).

This proves the following theorem.

**Theorem 2** (enumeration of second degree hypersurfaces of an  $n$ -dimensional projective space over an arbitrary field  $\mathbb{K}$ ). *Every second degree hypersurface in an  $n$ -dimensional projective space over a field  $\mathbb{K}$  of characteristic other than two is either an oval hypersurface or a  $k$ -fold ( $1 \leq k \leq n$ ) cylin-*

der over an oval hypersurface in an  $n - k$ -dimensional projective space.  $\square$

A one-dimensional projective space is a straight line and an oval "hypersurface" in it is a pair of distinct points (or an empty set). The corresponding  $n - 1$ -fold cylinder therefore is a pair of distinct hypersurfaces.

For  $k = n$  the situation is more intricate. A zero-dimensional projective space is a point and an oval hypersurface in it is an empty set. At the same time the equation  $x_0^2 = 0$  defines a "double" hyperplane  $x_0 = 0$  in an  $n$ -dimensional ( $n > 0$ ) projective space. To bring this case to common terminology therefore one has to assume that an  $n$ -fold cylinder over an empty set is a hyperplane in an  $n$ -dimensional space and that in a zero-dimensional projective space an oval second degree hypersurface is a "doubled" empty set.

It is also convenient to introduce the concept of a  $0$ -fold cylinder over a given hypersurface meaning by that cylinder the hypersurface itself. Then any second degree hypersurface in an  $n$ -dimensional projective space will be a  $k$ -fold ( $0 \leq k \leq n$ ) cylinder over some oval hypersurface in an  $n - k$ -dimensional space.

In the case  $\mathbb{K} = \mathbb{C}$  all the coefficients  $\lambda_1, \dots, \lambda_r$  of equation (1) may be assumed to be equal to unity. For any  $r$ ,  $0 \leq r \leq n$  therefore there is only one hypersurface (1) and by rank invariance these hypersurfaces are not projectively equivalent when  $r$  are different. This proves the following theorem.

**Theorem 3** (classification of second degree hypersurfaces of an  $n$ -dimensional projective space over the field  $\mathbb{C}$ ). *In an  $n$ -dimensional complex projective space there are only  $n + 1$  projectively non-Euclidean second degree hypersurfaces, one oval hypersurface and, for any  $r$ ,  $0 \leq r \leq n - 1$ , an  $(n - r)$ -fold cylinder over an oval hypersurface in an  $r$ -dimensional space.  $\square$*

In the case  $\mathbb{K} = \mathbb{R}$  the geometrical situation, as we know from [1], is not adequate to the algebraic one and one has to introduce *real-complex* spaces (i.e. to pass to the situation  $(\mathbb{C}, \mathbb{R})$ ; cf. Lecture 20 in [1]).

We stress that the algebraic situation remains unaffected in this case: all transformations of coordinates continue to



be transformations over  $\mathbb{R}$  and all equations have real coefficients.

■ A second degree hypersurface in a real-complex projective (or affine; see below) space is said to be *s-planar* ( $s \geq -1$ ) if the hypersurface contains no  $s + 1$ -dimensional plane but through any of its real points at least one (real)  $s$ -dimensional plane passes contained entirely in the hypersurface. In a three-dimensional space, for example, a hyperboloid of two sheets is 0-planar and a hyperboloid of one sheet is 1-planar. A hypersurface is 1-planar if and only if it contains no real points.

In the situation  $(\mathbb{C}, \mathbb{R})$  equation (1) can be reduced to the form

$$2) \quad x_0^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_r^2 = 0, \quad 0 \leq r \leq n,$$

it being possible owing to the multiplication of the equation by  $-1$  to assume without loss of generality that

$$-1 \leq p \leq \left[ \frac{r+1}{2} \right] - 1,$$

where  $\left[ \frac{r+1}{2} \right]$  is the integral part of the number  $\frac{r+1}{2}$  (if  $\left[ \frac{r+1}{2} \right] = m$ , then  $r = 2m$  or  $r = 2m - 1$ ).

When  $r = n$  the hypersurface (2) is called a *nonsingular second degree hypersurface*. It can be shown (do it!) that the nonsingular hypersurface (2) is  $p$ -planar. Thus, in particular, for  $p = -1$  the nonsingular hypersurface (2) has no real points. It is called an *imaginary oval second degree hypersurface*. When  $p = 0$  the nonsingular hypersurface (2) is called a *real oval second degree hypersurface*. When  $p > 1$  and  $r = n$  the hypersurface (2) is called in an unsophisticated way a *nonsingular  $p$ -planar second degree hypersurface*.

Since  $p$ -planarity is obviously a projectively invariant property, all nonsingular hypersurfaces (2) are projectively not equivalent.

When  $r < n$  the hypersurface (2) is an  $n - r$ -fold cylinder over a nonsingular hypersurface in an  $r$ -dimensional space, given by the same equation (2). Therefore all hypersurfaces (2) are projectively not equivalent either.

This proves the following theorem.

**Theorem 4** (classification of second degree hypersurfaces of a real-complex  $n$ -dimensional projective space). *In a real-complex  $n$ -dimensional ( $n > 0$ ) projective space there are only  $\left[\frac{n+1}{2}\right] + 1$  projectively non-equivalent, nonsingular second degree hypersurfaces that are not cylinders: two oval hypersurfaces (an imaginary and a real one) and (when  $n > 2$ ) one  $p$ -planar hypersurface for every  $p = 1, \dots, \left[\frac{n+1}{2}\right] - 1$ .*

*All the other second degree hypersurfaces are  $k$ -fold ( $1 \leq k \leq n$ ) cylinders over nonsingular hypersurfaces in an  $n - k$ -dimensional space (when  $k = n$ , they are double hyperplanes).  $\square$*

Similar theorems hold of course also in a *projective-affine space* obtained from a projective space by choosing some hyperplane as an ideal hyperplane. In such a space, second degree hypersurfaces will in addition differ in their positions relative to the ideal hyperplane. For example, instead of single  $k$ -fold cone cylinders there arise two classes of hypersurfaces: cylinders, if the axial plane  $\Pi_0$  is contained entirely in the ideal hyperplane,  $(k - 1)$ -fold cylinders over cones, if the plane  $\Pi_0$  has proper points (in the case where  $\Pi_0$  is a proper point there occur simply cones). Therefore the classification of second degree hypersurfaces even in a complex projective-affine space, trivial as it is, is rather awkward. That is why we shall not even formulate corresponding theorems.

On removing from the projective-affine space the ideal hyperplane we obtain an affine space. Therefore a classification of second degree hypersurfaces in a complex affine space can be obtained from their classification in a projective-affine space, the number of classes becoming only smaller. To attain a greater geometrical clarity, however, we prefer to obtain this classification directly.

Let  $\mathcal{A}$  be an affine  $n$ -dimensional space (over a yet arbitrary field  $\mathbb{K}$  of characteristic other than two) and let  $\mathcal{V}$  be an associated vector space.

**Definition 3.** *A second degree hypersurface in an affine space  $\mathcal{A}$  is a subset of the space, consisting of points whose*

affine coordinates  $x_1, \dots, x_n$  satisfy an equation of the form

$$F(x_1, \dots, x_n) = 0,$$

where  $F(x_1, \dots, x_n)$  is some second degree polynomial in  $x_1, \dots, x_n$ . Cf. Definition 2 of Lecture 18 in [1].

By introducing the vector  $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_n\mathbf{e}_n$  (i.e. the radius vector of a point  $M(x_1, \dots, x_n)$ ) we can write the equation  $F(x_1, \dots, x_n) = 0$  in the following "vector" form:

$$(3) \quad A(\mathbf{x}) + 2\alpha(\mathbf{x}) + a_{00} = 0,$$

where  $A$  is some quadratic functional:

$$A(\mathbf{x}) = \sum_{i,j=1}^n a_{ij} x_i x_j,$$

$\alpha$  is some linear functional:

$$\alpha(\mathbf{x}) = \sum_{i=1}^n a_{i0} x_i$$

and  $a_{00}$  is some number. (According to the notation adopted in the first semester, it would be necessary to write the index  $n+1$  instead of the index 0, but for the sake of simplicity we prefer to change the notation.)

By translating the origin of coordinates  $O$  into a point  $O'$  we obtain for each point  $M \in \mathcal{A}$  a new radius vector  $\mathbf{x}' = \overrightarrow{O'M}$  connected with the previous radius vector  $\mathbf{x} = \overrightarrow{OM}$  by the relation

$$\mathbf{x} = \mathbf{x}' + \mathbf{x}_0,$$

where  $\mathbf{x}_0 = \overrightarrow{OO'}$ . Therefore equation (3) is replaced by the equation

$$A(\mathbf{x}' + \mathbf{x}_0) + 2\alpha'(\mathbf{x}' + \mathbf{x}_0) + a_{00} = 0,$$

i.e. (we drop the prime in the notation for the vector  $\mathbf{x}'$ ) by the equation

$$A'(\mathbf{x}) + 2\alpha'(\mathbf{x}) + a'_{00} = 0,$$

where

$$(4) \quad \begin{aligned} A' &= A, \\ \alpha' &= \alpha_0 + \alpha, \\ a'_{00} &= A(\mathbf{x}_0) + 2\alpha(\mathbf{x}_0) + a_{00}. \end{aligned}$$

(Here the symbol  $\alpha_0$  denotes the associated covector  $x \mapsto A(\mathbf{x}, \mathbf{x}_0)$ .)

**Definition 4.** A point with radius vector  $\mathbf{x}_0 = x_1^{(0)}\mathbf{e}_1 + \dots + x_n^{(0)}\mathbf{e}_n$  is said to be a *centre* of hypersurface (3) if

$$\alpha_0 + \alpha = 0,$$

i.e. if

$$(5) \quad \sum_{j=1}^n a_{ij}x_j^{(0)} + a_{i0} = 0, \quad i = 1, \dots, n.$$

Relations (5) constitute a system of  $n$  equations in  $n$  unknowns  $x_1^{(0)}, \dots, x_n^{(0)}$ . If the system has a unique solution, i.e. if there exists a unique centre, then the hypersurface (3) is said to be *central*, otherwise it is said to be *noncentral*.

The determinant of system (5) is the determinant

$$(6) \quad \delta = \begin{vmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{vmatrix}$$

of the matrix of a functional  $A$ . Therefore, if  $\delta \neq 0$ , then according to Cramer's rule system (5) has a unique solution. If, however,  $\delta = 0$ , then the matrix rank  $r$  of system (5) is less than  $n$  and therefore (the Capelli-Kronecker theorem) system (5) is either incompatible (there are no centres), which is the case when the rank of the matrix

$$(7) \quad \begin{pmatrix} a_{10} & a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{nn} \end{pmatrix}$$

is equal to  $r + 1$ , or defines in the space  $\mathcal{A}$  a plane (a *plane of centres*) of dimension  $n - r$ , when the rank of the matrix (7) is equal to  $r$ .

Thus the *hypersurface* (3) is *central* if and only if  $\delta \neq 0$ .  $\square$

If the hypersurface (3) has at least one centre, then by translating the origin of coordinates into the centre we obtain for it an equation of the form

$$A(\mathbf{x}) + a_{00} = 0.$$

If  $a_{00} \neq 0$ , then we may divide the equation by  $a_{00}$  without loss of generality. In addition, according to the Lagrange theorem we may choose a basis of the coordinate system so that we have

$$(8) \quad A(\mathbf{x}) = \lambda_1 x_1^2 + \dots + \lambda_r x_r^2,$$

where  $\lambda_1 \neq 0, \dots, \lambda_r \neq 0$ . This proves that for a second degree hypersurface with a centre there exists a system of affine coordinates  $x_1, \dots, x_n$  in which its equation has the form

$$\lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = \varepsilon,$$

where  $\lambda_1 \neq 0, \dots, \lambda_r \neq 0$  and  $\varepsilon = 0, 1$ .  $\square$

When  $r = n$  the hypersurface is central and when  $r < n$  it is an  $n - r$ -fold cylinder over a central hypersurface in an  $r$ -dimensional space.

Suppose now that the hypersurface (3) has no centre (which, we remark, is possible only when  $n > 1$ ). This means that in the conjugate space  $\mathcal{V}'$  the covector  $\alpha$  is not of the form  $-\alpha_0$ , i.e. is not a covector associated with the bilinear functional  $A$  and hence is not in the rank space  $\mathcal{R}$  of the functional  $A$ .

We reduce the quadratic functional  $A$  to the form (8). This means that in the rank space  $\mathcal{R}$  of the corresponding bilinear functional  $A$  we find a basis  $\mathbf{e}^1, \dots, \mathbf{e}^r$  such that

$$A = \lambda_1 \mathbf{e}^1 \otimes \mathbf{e}^1 + \dots + \lambda_r \mathbf{e}^r \otimes \mathbf{e}^r.$$

(To obtain a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  in which the values of the quadratic functional  $A$  are expressed by formula (8) one should extend this basis to a basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  of the space  $\mathcal{V}'$  and change to the conjugate basis.)

Since  $\alpha \notin \mathcal{R}$ , it is clear that we may choose a basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  so that we have  $\mathbf{e}^{r+1} = -\alpha$ , i.e. so that  $\alpha(\mathbf{x}) = -x_{r+1}$  for any vector  $\mathbf{x} \in \mathcal{V}$ .

In such a basis, for any initial point  $O$  an equation of the form (3) becomes

$$(9) \quad \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 - 2x_{r+1} + a_{00} = 0.$$

By translating the origin of coordinates  $O$  into a point with coordinates

$$\left( \underbrace{0, \dots, 0}_{r \text{ times}}, \frac{a_{00}}{2}, 0, \dots, 0 \right),$$

we obviously (see the last of the formulas (4)) obtain an equation of the form (9) with  $a_{00} = 0$ .

This proves the following theorem.

**Theorem 5 (reduction of the equations of second degree hypersurfaces in an  $n$ -dimensional affine space over an arbitrary field  $\mathbb{K}$  to normal form).** *For any second degree hypersurface in an  $n$ -dimensional affine space over a field  $\mathbb{K}$  of characteristic other than two there exists a system of affine coordinates in which its equation has either the form*

$$(I) \quad \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = \varepsilon,$$

where  $1 \leq r \leq n$  and  $\varepsilon = 0$  or  $1$ , or (which is possible only when  $n > 1$ ) the form

$$(II) \quad \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = 2x_{r+1},$$

where  $1 \leq r \leq n - 1$ , with  $\lambda_1 \neq 0, \dots, \lambda_r \neq 0$  in both cases.  $\square$

When  $r = n$  and  $\varepsilon = 1$  the hypersurface (I) is called an *oval second degree hypersurface*. When  $r = n$  and  $\varepsilon = 0$  it is called a *second degree cone* and is a cone over an oval hypersurface in an  $n - 1$ -dimensional space. When  $r < n$  the hypersurface (I) is an  $n - r$ -fold cylinder whose base is either an oval hypersurface (when  $\varepsilon = 1$ ) or a second degree cone (when  $\varepsilon = 0$ ) in an  $r$ -dimensional space.

The hypersurface (II) is called a *paraboloid*, when  $r = n - 1$ . When  $r < n - 1$  it is an  $n - r - 1$ -fold cylinder over a paraboloid in an  $r + 1$ -dimensional space.

Thus the following theorem holds.

**Theorem 6 (enumeration of second degree hypersurfaces in an  $n$ -dimensional affine space over an arbitrary field  $\mathbb{K}$ ).**

*Every second degree hypersurface in an  $n$ -dimensional affine space over a field  $\mathbb{K}$  of characteristic other than two is either*

- (a) an oval hypersurface or*
- (b) a cone or*
- (c) a paraboloid (when  $n > 1$ ) or*
- (d) a  $k$ -fold cylinder,  $1 \leq k \leq n - 1$ , over one of the hypersurfaces of types (a), (b), (c) in an  $n - k$ -dimensional affine space.*

*Hypersurfaces of different types are affinely not equivalent.*

The last statement follows from the fact that

(i) hypersurfaces of type (b) possess a *vertex* (a point for which the straight line connecting it to an arbitrary point of a hypersurface lies entirely on that hypersurface) while those of type (a) do not;

(ii) hypersurfaces of types (a) and (b) have a *centre of symmetry* while those of type (c) have not;

(iii) hypersurfaces of type (d) are cylinders while those of types (a), (b) and (c) are not.  $\square$

When  $\mathbb{K} = \mathbb{C}$  there is, up to affine equivalence, only one second degree hypersurface in each of the classes (a), (b), and (c). This means that the following theorem is true:

**Theorem 7 (classification of second degree hypersurfaces of an  $n$ -dimensional affine space over the field  $\mathbb{C}$ ).** *In an  $n$ -dimensional complex affine space there are only two affinely nonequivalent second degree hypersurfaces for  $n = 1$ : an oval hypersurface consisting of two distinct points and a second degree cone representing two coincident points, and for  $n > 1$  there are three such hypersurfaces that are not cylinders: an oval hypersurface, a second degree cone, and a paraboloid. The other second degree hypersurfaces in an  $n$ -dimensional ( $n > 1$ ) affine space are  $k$ -fold ( $1 \leq k \leq n - 1$ ) cylinders over the three (two for  $k = n - 1$ ) indicated hypersurfaces in an  $n - k$ -dimensional affine space.  $\square$*

When  $\mathbb{K} = \mathbb{R}$  (in the situation  $(\mathbb{C}, \mathbb{R})$ ) equation (I) can be reduced to the form

$$(I') \quad x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_r^2 = \varepsilon,$$

where  $\varepsilon = -1, 0$  or  $1$  and  $1 \leq r \leq n$ , and equation (II) to the form

$$(II') \quad x_1^2 + \dots + x_p^2 - x_{p+1}^2 - \dots - x_r^2 = 2x_{r+1},$$

where  $1 \leq r \leq n - 1$ , it being possible, owing to the multiplication by  $-1$  (and the change of the sign in the coordinate  $x_{r+1}$ ), to assume without loss of generality in both cases that  $0 \leq p \leq \frac{n}{2}$  (and in case (I'), with  $p = \frac{n}{2}$  and hence with  $n$  even, also, in addition, that  $\varepsilon \neq -1$ ).

When  $r = n$  the hypersurface (I') is called a *nonsingular second degree hypersurface*. When  $\varepsilon \neq 0$  and  $p = 0$  the nonsingular hypersurface is called an *ellipsoid*, an *imaginary* one if  $\varepsilon > 0$  and a *real* one if  $\varepsilon < 0$ . When  $n = 2$  we have an imaginary and a real ellipsoid, and when  $n = 1$  we have pairs of imaginary or real points.

When  $\varepsilon = 0$  the nonsingular second degree hypersurface is called a *second degree cone*. When  $p = 0$  the second degree cone contains only one real point and for this reason it is usually called an *imaginary second degree cone*.

When  $\varepsilon \neq 0$  and  $1 \leq p \leq \frac{n}{2}$  the nonsingular second degree hypersurface is called an  $\varepsilon$ -hyperboloid.

When  $n = 2$  there exists only one hyperboloid—a hyperbola and two cones—pairs of imaginary and real intersecting straight lines. When  $n = 1$ , there are no hyperboloids and there is only one cone—a pair of coincident points.

Just as in the projective space the second degree hypersurface in a real-complex affine space is said to be *s-planar* if at least one  $s$ -dimensional plane lying entirely in the hypersurface passes through any of its real points, but no  $s + 1$ -dimensional plane is contained in the hypersurface.

It can be shown (do it!) that every  $\varepsilon$ -hyperboloid is  $s$ -planar, where  $s = p - 1$ , if  $\varepsilon = 1$ , and  $s = p$ , if  $\varepsilon = -1$  and that every second degree cone is  $p$ -planar.

When  $r < n$  hypersurfaces (I') are  $n - r$ -fold cylinders over nonsingular hypersurfaces in an  $r$ -dimensional space.

When  $r = n - 1$  the hypersurface (II') is called a *paraboloid*, an *elliptical* one if  $p = 0$  (for  $n = 2$  it is a parabola) and a *hyperbolic* one if  $1 \leq p \leq \frac{n}{2}$ . It can be shown (do it!) that every paraboloid is  $p$ -planar.

When  $r < n - 1$  the hypersurface (II') is an  $n - r - 1$ -fold cylinder over a parabola in an  $r$ -dimensional space.



As in the case  $\mathbb{K} = \mathbb{C}$ , it is proved that ellipsoids together with hyperboloids, as well as cones, paraboloids and cylinders are affinely not equivalent. Paraboloids are affinely not equivalent, for they are  $p$ -planar when  $p$  are different. For the same reason, neither are cones, nor  $\varepsilon$ -hyperboloids with the same  $\varepsilon$ . The real and imaginary ellipsoids are in the obvious way affinely not equivalent to each other, nor are they to any  $\varepsilon$ -hyperboloid, with a possible exception of the 1-hyperboloid with  $p = 1$  (i.e. the 0-planar one). But there are hyperbolas among the sections of the latter hyperboloid by two-dimensional planes, which is not true for the ellipsoid. Therefore the ellipsoid and 0-planar 1-hyperboloid are affinely not equivalent either. Finally, when  $s > 1$ , for the  $s$ -planar 1-hyperboloid (corresponding to the value  $p = s + 1$ ) the maximum dimension of planes cutting it in an imaginary ellipsoid (i.e. not intersecting it in the real domain) equals (prove it!)  $n - s - 1 = n - p$  and for the  $s$ -planar  $-1$ -hyperboloid (for which  $p = s$ ) a similar dimension equals (prove it!)  $s = p$ . Since in this situation the equation  $p = n - p$  is impossible (for when  $n = 2p$  the case  $\varepsilon = -1$  is excluded under the hypothesis), we see that the  $s$ -planar  $\pm 1$ -hyperboloids are affinely not equivalent either.

This proves the following theorem.

**Theorem 8 (classification of second degree hypersurfaces of an  $n$ -dimensional affine space in the situation  $(\mathbb{C}, \mathbb{R})$ ).** *In the  $n$ -dimensional real-complex affine space there are only the following affinely nonequivalent second degree hypersurfaces that are not cylinders:*

- (a) *two ellipsoids (an imaginary and a real one);*
- (b) *one  $s$ -planar 1-hyperboloid for any  $s = 0, 1, \dots$*   
 $\dots, \left\lfloor \frac{n}{2} \right\rfloor - 1;$
- (c) *one  $s$ -planar  $-1$ -hyperboloid for any  $s = 1, \dots, m$ ,*  
*where  $m = \frac{n}{2} - 1$ , if  $n$  is even, and  $m = \frac{n-1}{2}$ , if  $n$  is odd;*
- (d) *one  $p$ -planar second degree cone for any  $p = 0, 1, \dots$*   
 $\dots, \left\lfloor \frac{n}{2} \right\rfloor$  *(for  $p = 0$  it is an imaginary cone);*
- (e) *one  $p$ -planar paraboloid for any  $p = 0, \dots, \left\lfloor \frac{n}{2} \right\rfloor$  (for*

$p = 0$  it is an elliptical paraboloid and for  $p = 1, \dots$   
 $\dots, \left[ \frac{n}{2} \right]$  we have hyperbolic paraboloids).

All the other second degree hypersurfaces are  $k$ -fold cylinders ( $1 \leq k \leq n - 1$ ) over the enumerated hypersurfaces in an  $n - k$ -dimensional affine space.  $\square$

# Lecture 14

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*The algebra of linear operators • Operators and mixed bilinear functionals • Linear operators and matrices • Invertible operators • The adjoint operator • The Fredholm alternative • Invariant subspaces and induced operators*

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Let us now return to the theory of vector spaces and consider the last type of bilinear functionals which we have not studied yet, mixed functionals  $B: \mathbf{x}, \xi \mapsto B(\mathbf{x}, \xi)$ , where  $\mathbf{x} \in \mathcal{V}$ ,  $\xi \in \mathcal{V}'$  (see Lecture 5). It turns out that these functionals are closely related to homomorphisms (see Definition 5 of Lecture 3) for which  $\mathcal{W} = \mathcal{V}$ .

**Definition 1.** Homomorphisms from  $\mathcal{V}$  into  $\mathcal{V}$  are *linear operators on  $\mathcal{V}$* .

Thus the mapping

$$(1) \quad \mathbf{A}: \mathcal{V} \rightarrow \mathcal{V}$$

is a linear operator if

$$\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y}$$

and

$$\mathbf{A}(k\mathbf{x}) = k\mathbf{A}\mathbf{x}$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  and any number  $k \in \mathbb{K}$ .

The *sum*  $\mathbf{A} + \mathbf{B}$  of linear operators  $\mathbf{A}$  and  $\mathbf{B}$  and the *product*  $k\mathbf{A}$  of a linear operator  $\mathbf{A}$  by a number  $k \in \mathbb{K}$  are defined in the usual way:

$$\begin{aligned} (\mathbf{A} + \mathbf{B}) \mathbf{x} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{x}, \\ (k\mathbf{A}) \mathbf{x} &= k(\mathbf{A}\mathbf{x}), \end{aligned}$$

and are obviously linear operators. It can be immediately verified that under these operations the set  $\text{Op } (\mathcal{V})$  of all linear operators on  $\mathcal{V}$  is a vector space.

Serving as the zero of that space is a zero operator  $\mathbf{O}$  acting according to the formula

$$\mathbf{O} (\mathbf{x}) = \mathbf{0}.$$

For operators the multiplication  $\mathbf{A}, \mathbf{B} \mapsto \mathbf{AB}$ , where, as is usual for mappings, the composition  $\mathbf{A} \circ \mathbf{B}$  of operators is regarded as their product  $\mathbf{AB}$ , is defined as well as addition. Thus

$$(\mathbf{AB}) \mathbf{x} = \mathbf{A} (\mathbf{Bx})$$

for any vector  $\mathbf{x} \in \mathcal{V}$ . The operator  $\mathbf{AB}$  is obviously linear.

A trivial calculation shows that multiplication of operators is associative:

$$(\mathbf{AB}) \mathbf{C} = \mathbf{A} (\mathbf{BC})$$

(so that it is possible not to write parentheses in the product of any number of operators) and distributive over addition:

$$\mathbf{A} (\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}.$$

This means that the set  $\text{Op } (\mathcal{V})$  is also a ring.

The ring possesses unity which is an identity operator  $\mathbf{E}$ :  $\mathcal{V} \rightarrow \mathcal{V}$  leaving every vector  $\mathbf{x} \in \mathcal{V}$  fixed:

$$\mathbf{Ex} = \mathbf{x}.$$

In general  $\mathbf{AB} \neq \mathbf{BA}$ , so that the ring  $\text{Op } (\mathcal{V})$  is noncommutative (for  $n > 1$ ).

Multiplication of operators is related to their multiplication by numbers  $k \in \mathbb{K}$  by the formula

$$(2) \quad (k\mathbf{A}) \mathbf{B} = \mathbf{A} (k\mathbf{B}) = k (\mathbf{AB})$$

whose proof reduces to a trivial calculation.

Rings which are at the same time vector spaces and in which relation (2) holds are called *algebras*. Thus, summing up all the foregoing we see that the set  $\text{Op } (\mathcal{V})$  is an algebra.  $\square$

From relation (2) it follows in particular that

$$(k\mathbf{E}) \mathbf{A} = \mathbf{A} (k\mathbf{E})$$

for any operator  $A$ . Thus operators of the form  $kE$ , called *scalar operators*, are commutative with all operators.

It turns out (try to show this on your own) that this property characterizes scalar operators, i.e. any operator commutative with every operator of  $\text{Op}(\mathcal{V})$  is scalar. The algebra  $\text{Op}(\mathcal{V})$  can thus be said to be noncommutative to a maximum extent (to an extent permitted by the structure of the algebra).

Every operator  $A$  defines according to the formula

$$A(\mathbf{x}, \xi) = \xi(A\mathbf{x})$$

some mixed bilinear functional  $A \in T_1^1(\mathcal{V})$ . Conversely, for any mixed bilinear functional  $A$  the correspondence assigning to an arbitrary vector  $\mathbf{x} \in \mathcal{V}$  an associated covector

$$A\mathbf{x}: \xi \mapsto A(\mathbf{x}, \xi)$$

of a space  $\mathcal{V}'$  (i.e., by virtue of the identification  $(\mathcal{V}')' = \mathcal{V}$ , a vector of the space  $\mathcal{V}$ ) is a linear operator  $A \in \text{Op}(\mathcal{V})$ . Since the constructed mappings  $A \mapsto A$  and  $A \mapsto A$  are obviously reciprocal, each is bijective. Since these mappings obviously carry a sum over into a sum and a product by a number into a product by the same number, they are both isomorphisms. This proves that the *vector spaces*  $\text{Op}(\mathcal{V})$  and  $T_1^1(\mathcal{V})$  are *isomorphic in a natural way*.  $\square$

As a rule we shall identify an operator with the corresponding bilinear functional.

Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be some basis chosen in a space  $\mathcal{V}$ . Then for any vector  $\mathbf{x} = x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n$  we have

$$(3) \quad A\mathbf{x} = x^1\mathbf{a}_1 + \dots + x^n\mathbf{a}_n,$$

where  $\mathbf{a}_1 = A\mathbf{e}_1, \dots, \mathbf{a}_n = A\mathbf{e}_n$ . Conversely, for any family of vectors  $\mathbf{a}_1, \dots, \mathbf{a}_n$  (3) uniquely defines some linear operator  $A$  for which  $\mathbf{a}_1 = A\mathbf{e}_1, \dots, \mathbf{a}_n = A\mathbf{e}_n$ . Thus, with the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  fixed, the operators  $A \in \text{Op}(\mathcal{V})$  are in *bijective correspondence with  $n$ -member families of vectors  $\mathbf{a}_1, \dots, \mathbf{a}_n$* .  $\square$

To every such family there corresponds a quadratic matrix whose *columns* consist of the coordinates of vectors

$\mathbf{a}_1, \dots, \mathbf{a}_n$  (in the same basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ ):

$$(4) \quad A = \begin{pmatrix} a_1^1 & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \dots & a_n^n \end{pmatrix}.$$

Since this obviously establishes a bijective correspondence between matrices and families  $\mathbf{a}_1, \dots, \mathbf{a}_n$  of vectors, we thus obtain a bijective correspondence between operators and quadratic matrices of order  $n$ . An automatic computation verifies that this correspondence is an isomorphism (carries a sum over into a sum and a product by a number into a product by the same number).

Thus we have proved the following proposition.

**Proposition 1.** *The choice of a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of an  $n$ -dimensional vector space  $\mathcal{V}$  over a field  $\mathbb{K}$  establishes an isomorphism between the algebra of operators  $\text{Op}(\mathcal{V})$  and the algebra of quadratic matrices of order  $n$  over  $\mathbb{K}$ .  $\square$*

Corresponding to an operator  $A$  under this isomorphism is a matrix  $A$  whose columns consist of the coordinates of vectors  $A\mathbf{e}_1, \dots, A\mathbf{e}_n$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .

**Definition 2.** The matrix  $A$  is called the *matrix of the operator  $A$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$* .

Since  $a_i^j = \mathbf{e}^j(A\mathbf{e}_i)$ , we see that  $A$  is simultaneously the matrix of a mixed bilinear functional  $A$ . It follows (see Lecture 5) that the matrix  $A' = (a_i^{j'})$  of the operator  $A$  in any other basis  $\mathbf{e}_{1'}, \dots, \mathbf{e}_{n'}$ , is expressed by the formula

$$(5) \quad A' = C^{-1}AC,$$

where  $C = (c_i^{j'})$  is the transition matrix.

However, (5) can be established without difficulty by direct computation: since  $\mathbf{e}_{i'} = c_i^{i'}\mathbf{e}_i$  and  $\mathbf{e}_j = c_j^{j'}\mathbf{e}_{j'}$ , we have  $a_i^{j'}\mathbf{e}_{j'} = A\mathbf{e}_{i'} = c_i^{i'}A\mathbf{e}_i = c_i^{i'}a_i^j\mathbf{e}_j = c_i^{i'}a_i^jc_j^{j'}\mathbf{e}_{j'}$ , and this is equivalent to (5). Of course this computation is in fact a repetition of the one in Lecture 5.

To carry out the same computation in matrix notation we introduce the vector row matrices

$$\begin{aligned} \mathbf{e} &= (\mathbf{e}_1, \dots, \mathbf{e}_n), & \mathbf{e}' &= (\mathbf{e}_{1'}, \dots, \mathbf{e}_{n'}), \\ A\mathbf{e} &= (A\mathbf{e}_1, \dots, A\mathbf{e}_n), & A\mathbf{e}' &= (A\mathbf{e}_{1'}, \dots, A\mathbf{e}_{n'}). \end{aligned}$$

Then (cf. formula (14) of Lecture 10 in [1])

$$e' = eC, \quad e = e'C^{-1}$$

and

$$Ae = eA, \quad Ae' = e'A'.$$

On the other hand, by linearity

$$Ae' = A(eC) = (Ae)C.$$

Therefore

$$e^1A' = Ae' = (Ae)C = eAC = e'C^{-1}AC,$$

and hence  $A' = C^{-1}AC$ .  $\square$

An operator  $A$  is said to be *nonsingular* if  $\det A \neq 0$  (and respectively *singular* if  $\det A = 0$ ). It follows immediately from formula (5) that this definition is correct.

Of particular interest are *invertible operators*, i.e. such operators for which there exists an *inverse operator*  $A^{-1}$  satisfying the relations

$$AA^{-1} = A^{-1}A = E.$$

The operator  $A$  is said to be *left invertible* if there exists an operator  $B$  such that

$$BA = E,$$

and *right invertible* if there exists an operator  $C$  such that

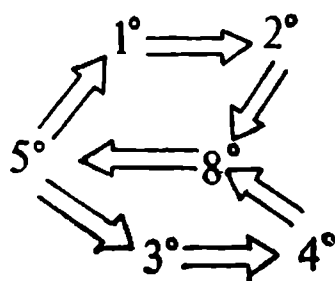
$$AC = E.$$

In arbitrary rings (or algebras) there exist invertible elements that are only right or only left invertible. For linear operators the situation is quite different, however; an operator is invertible if it is at least left or right invertible. This is closely related to the (truly remarkable) fact that a linear operator is bijective if it is merely injective or surjective. (We remark here that although an invertible operator is obviously bijective, the statement that any bijective linear operator is invertible, i.e. that invertible operator is linear, requires proof.)

**Proposition 2.** *For any linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  the following statements are equivalent.*

- 1° The operator  $A$  is left invertible.
- 2° The operator  $A$  is injective, i.e.  $\text{Ker } A = 0$ .
- 3° The operator  $A$  is right invertible.
- 4° The operator  $A$  is surjective, i.e.  $\text{Im } A = \mathcal{V}$ .
- 5° The operator  $A$  is invertible.
- 6° The operator  $A$  is bijective.
- 7° The operator  $A$  is nonsingular.
- 8° For any basis  $e_1, \dots, e_n$  of a space  $\mathcal{V}$  vectors  $Ae_1, \dots, Ae_n$  also constitute a basis.

*Proof.* The equivalence of statements 7° and 8° follows immediately from the matrix rank theorem. It is therefore necessary to prove only the equivalence of statements 1° to 6° and 8°. To do this it is sufficient to prove the following diagram of implications:



*Implication  $5^\circ \Rightarrow 1^\circ$ .* If  $A^{-1}$  is an inverse operator, then  $A^{-1}A = E$ .

*Implication  $1^\circ \Rightarrow 2^\circ$ .* If  $BA = E$  and  $Ax = 0$ , then  $x = Ex = BAx = B0 = 0$ .

*Implication  $2^\circ \Rightarrow 8^\circ$ .* If the vectors  $Ae_1, \dots, Ae_n$  are linearly dependent, i.e.  $k_1Ae_1 + \dots + k_nAe_n = 0$ , where  $(k_1, \dots, k_n) \neq (0, \dots, 0)$ , then for a vector  $e = k_1e_1 + \dots + k_ne_n \neq 0$  we have  $Ae = 0$ . Consequently, if  $\text{Ker } A = 0$ , then the vectors  $Ae_1, \dots, Ae_n$  are linearly independent and hence constitute a basis.

*Implication  $5^\circ \Rightarrow 3^\circ$ .* If  $A^{-1}$  is an inverse operator, then  $AA^{-1} = E$ .

*Implication  $3^\circ \Rightarrow 4^\circ$ .* If  $AC = E$ , then  $Ay = x$  for any vector  $x \in \mathcal{V}$ , where  $y = Cx$ .

*Implication  $4^\circ \Rightarrow 8^\circ$ .* If for any vector  $x \in \mathcal{V}$  there exists a vector  $y \in \mathcal{V}$  such that  $Ay = x$ , then  $x = y^1Ae_1 + \dots + y^nAe_n$ . This proves that the family  $Ae_1, \dots, Ae_n$  consisting of  $n$  vectors is complete. Hence it is a basis.



*Implication*  $8^\circ \Rightarrow 5^\circ$ . In the basis  $\mathbf{e}'_1 = \mathbf{A}\mathbf{e}_1, \dots, \mathbf{e}'_n = \mathbf{A}\mathbf{e}_n$  the family of vectors  $\mathbf{b}_1 = \mathbf{e}_1, \dots, \mathbf{b}_n = \mathbf{e}_n$  determines an operator  $\mathbf{B}$  for which  $\mathbf{B}\mathbf{e}'_1 = \mathbf{b}_1, \dots, \mathbf{B}\mathbf{e}'_n = \mathbf{b}_n$  and hence  $(\mathbf{B}\mathbf{A})\mathbf{e}_1 = \mathbf{e}_1, \dots, (\mathbf{B}\mathbf{A})\mathbf{e}_n = \mathbf{e}_n$ , i.e.  $\mathbf{B}\mathbf{A} = \mathbf{E}$ . For the same operator  $(\mathbf{A}\mathbf{B})\mathbf{e}'_1 = \mathbf{e}'_1, \dots, (\mathbf{A}\mathbf{B})\mathbf{e}'_n = \mathbf{e}'_n$  and hence  $\mathbf{A}\mathbf{B} = \mathbf{E}$ . Consequently the operator  $\mathbf{A}$  is invertible (and  $\mathbf{B} = \mathbf{A}^{-1}$ ).  $\square$

The vector equation

$$\mathbf{A}\mathbf{x} = \mathbf{b}$$

can be written in coordinates as a system of  $n$  linear equations in  $n$  unknowns. In terms of equations therefore the equivalence of statements  $2^\circ$  and  $4^\circ$  means that a *system of  $n$  nonhomogeneous linear equations in  $n$  unknowns is compatible for any free terms if and only if the corresponding system of homogeneous linear equations has only a trivial solution.*

A direct extension of this beautiful statement to the case where the number of equations is not equal to that of unknowns is shown by elementary examples to be false. To obtain such an extension it is first necessary to appropriately reformulate the statement.

Let  $\mathbf{A} \in \text{Op}(\mathcal{V})$ . We associate with an arbitrary covector  $\xi \in \mathcal{V}'$  a functional  $\mathbf{A}'\xi$  in  $\mathcal{V}$  by setting

$$(6) \quad (\mathbf{A}'\xi)(\mathbf{x}) = \xi(\mathbf{A}\mathbf{x}), \quad \mathbf{x} \in \mathcal{V}.$$

An automatic check shows that

- (a) the functional  $\mathbf{A}'\xi$  is linear, i.e. is a covector of  $\mathcal{V}'$ ;
- (b) the resulting mapping  $\mathbf{A}': \mathcal{V}' \rightarrow \mathcal{V}'$  is linear, i.e.  $\mathbf{A}'$  is a linear operator.

**Definition 3.** The operator  $\mathbf{A}'$  is called an operator *adjoint* to the operator  $\mathbf{A}$ .

If we introduce a natural pairing  $\langle \mathbf{x}, \xi \rangle = \xi(\mathbf{x})$  between spaces  $\mathcal{V}$  and  $\mathcal{V}'$  (see Lecture 4), then formula (6) defining the adjoint operator  $\mathbf{A}'$  takes the form

$$\langle \mathbf{x}, \mathbf{A}'\xi \rangle = \langle \mathbf{A}\mathbf{x}, \xi \rangle.$$

From the symmetry of the formula it immediately ensues that the mapping  $\mathbf{A} \mapsto \mathbf{A}'$  of the space  $\text{Op}(\mathcal{V})$  into a space  $\text{Op}(\mathcal{V}')$  is *involutory*, i.e.

$$\mathbf{A}'' = \mathbf{A}.$$

In particular it follows that the mapping  $A \mapsto A'$  is bijective.

Moreover, it is clear that

$$(A + B)' = A' + B' \text{ and } (kA)' = kA'.$$

This means that the mapping  $A \mapsto A'$  is an isomorphism of the vector space  $\text{Op}(\mathcal{V})$  into the vector space  $\text{Op}(\mathcal{V}')$ .  $\square$

There is thus no natural isomorphism between vector spaces  $\mathcal{V}$  and  $\mathcal{V}'$  but there is between the vector spaces  $\text{Op}(\mathcal{V})$  and  $\text{Op}(\mathcal{V}')$ !

With respect to multiplication, the mapping  $A \mapsto A'$  is not an isomorphism, since the order of cofactors is not changed:

$$(AB)' = B'A'.$$

Indeed,  $\langle \mathbf{x}, (AB)' \xi \rangle = \langle AB\mathbf{x}, \xi \rangle = \langle B\mathbf{x}, A'\xi \rangle = \langle \mathbf{x}, B'A'\xi \rangle$ . A linear isomorphism having this property is usually called an *anti-isomorphism*.

The formula  $a_i^j = e^j(Ae_i)$  for the elements of the matrix of the operator  $A$  in the basis  $e_1, \dots, e_n$  implies that

$$a_i^j = \langle Ae_i, e^j \rangle.$$

For the elements  $a_i'^j$  of the matrix of the adjoint operator  $A'$  in the conjugate basis  $e^1, \dots, e^n$  we therefore have

$$a_i'^j = \langle e_i, A'e^j \rangle = \langle Ae_i, e^j \rangle$$

and therefore  $a_i'^j = a_i^j$ , i.e.  $Ae^j = a_i^j e^i$ . This does not mean, however, that the matrices of the operators  $A$  and  $A'$  coincide. Indeed, by definition, the columns of the matrix of an operator are the coordinates of vectors resulting from the application of the operator to the vectors of the basis. For the operator  $A$  this means (by virtue of the formula  $Ae_i = a_i^j e_j$ ) that the  $i$ th column of its matrix consists of the numbers  $a_i^1, \dots, a_i^n$ . As to the operator  $A'$ , however, the formula  $Ae^j = a_i^j e^i$  implies that the  $j$ th column of its matrix consists of numbers  $a_1^j, \dots, a_n^j$ , i.e. of the same numbers that the  $j$ th row in the matrix of the operator  $A$  contains. Thus the matrix of the adjoint operator  $A'$  in the conjugate basis  $e^1, \dots, e^n$  is a matrix  $A^\top$  resulting from *transposing* the matrix  $A$  of the operator  $A$  in the basis  $e_1, \dots, e_n$ .



statements. The analogue of Proposition 3 remains, which just gives the Fredholm alternative in the general form.

**Definition 4.** The subspace  $\mathcal{F}$  of a space  $\mathcal{V}$  is said to be *invariant* under the operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  if

$$Ax \in \mathcal{F} \text{ for any vector } x \in \mathcal{F}.$$

Defined in this case is the operator

$$A|_{\mathcal{F}} \in \mathcal{O}p(\mathcal{F}),$$

acting according to the formula

$$(A|_{\mathcal{F}})x = Ax, \quad x \in \mathcal{F},$$

where the vector  $Ax$  at the right is regarded as an element of the subspace  $\mathcal{F}$ .

The operator  $A|_{\mathcal{F}}$  is called a *restriction* of the operator  $A$  to the invariant subspace  $\mathcal{F}$ . It is also said to be *induced* by the operator  $A$ .

Since  $\dim \mathcal{F} < \dim \mathcal{V}$ , the operator  $A|_{\mathcal{F}}$  lends itself to study more easily than the operator  $A$ . At the same time, by studying it we can often obtain sufficiently much information also about the operator  $A$  itself.

Especially satisfactory is the situation in the case (unfortunately, not always holding) where there exists a second invariant subspace  $\mathcal{Q}$  complementary to the subspace  $\mathcal{F}$ , i.e. where the space  $\mathcal{V}$  is the direct sum  $\mathcal{V} = \mathcal{F} \oplus \mathcal{Q}$  of the invariant subspaces  $\mathcal{F}$  and  $\mathcal{Q}$ . In this case the operator  $A$  can be completely determined by the operators  $A|_{\mathcal{F}}$  and  $A|_{\mathcal{Q}}$ . Indeed, for any vector  $z = x + y$  of a space  $\mathcal{V}$ , where  $x \in \mathcal{F}$ ,  $y \in \mathcal{Q}$ , we obviously have

$$Az = (A|_{\mathcal{F}})x + (A|_{\mathcal{Q}})y.$$

A complete reducibility of the operator  $A$  to the operators  $A|_{\mathcal{F}}$  and  $A|_{\mathcal{Q}}$  is clearly demonstrated by the matrix  $A = (a_i^j)$  of the operator  $A$  in a basis  $e_1, \dots, e_n$  of the space  $\mathcal{V}$  such that  $\mathcal{F} = [e_1, \dots, e_p]$  and  $\mathcal{Q} = [e_{p+1}, \dots, e_n]$ . Indeed, since  $Ae_i = a_i^j e_j \in \mathcal{F}$  for  $1 \leq i \leq p$ , we have

$$a_i^j = 0 \text{ if } 1 \leq i \leq p \text{ and } p+1 \leq j \leq n,$$

Similarly, since  $Ae_i \in \mathcal{Q}$  for  $p + 1 \leq i \leq n$ , we have

$$a_i^j = 0 \text{ if } p + 1 \leq i \leq n \text{ and } 1 \leq j \leq p.$$

This means that the matrix  $A$  has a *diagonal block form* in the basis  $e_1, \dots, e_n$ :

$$(9) \quad A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where  $A_1$  is the matrix of the operator  $A|_{\mathcal{P}}$  in the basis  $e_1, \dots, e_p$  and  $A_2$  is the matrix of the operator  $A|_{\mathcal{Q}}$  in the basis  $e_{p+1}, \dots, e_n$ .

A matrix  $A$  of the form (9) is sometimes said to be *decomposed as a direct sum of matrices  $A_1$  and  $A_2$*  (written  $A = A_1 \oplus A_2$ ). Thus every decomposition of a space  $\mathcal{V}$  as a direct sum of invariant subspaces determines a decomposition of the matrix of the operator as a direct sum of the matrices of induced operators.

In the case where the invariant subspace  $\mathcal{P}$  has no invariant complement  $\mathcal{Q}$  (or the latter is not known) we can represent the matrix  $A$  (by choosing a basis  $e_1, \dots, e_n$  so that  $\mathcal{P} = [e_1, \dots, e_p]$ ) in *triangular block form*

$$(10) \quad A = \begin{pmatrix} A_1 & C \\ 0 & B \end{pmatrix},$$

where  $A_1$  is the matrix of the operator  $A|_{\mathcal{P}}$ .

From the fact that the subspace  $\mathcal{P}$  is invariant under the operator  $A$  we immediately see that the formula

$$B(x + \mathcal{P}) = Ax + \mathcal{P}$$

correctly defines in the factor space  $\mathcal{V}/\mathcal{P}$  some (obviously linear) operator

$$B: \mathcal{V}/\mathcal{P} \rightarrow \mathcal{V}/\mathcal{P}.$$

The operator  $B$  is also said to be *induced* by the operator  $A$ .

If the basis  $e_1, \dots, e_n$  of the space  $\mathcal{V}$  is chosen so that  $\mathcal{P} = [e_1, \dots, e_p]$ , then the cosets  $e_{p+1} + \mathcal{P}, \dots, e_n + \mathcal{P}$  will obviously constitute a basis of the factor space  $\mathcal{V}/\mathcal{P}$  and the matrix of the operator  $B$  in that basis will be the matrix  $B$  of (10).

# Lecture 15

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*Eigenvalues • Characteristic roots • Diagonalizable operators • Operators with simple spectrum • The existence of a basis in which the matrix of an operator is triangular • Nilpotent operators*

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The simplest invariant subspaces are one-dimensional subspaces.

**Definition 1.** A vector  $\mathbf{x} \neq 0$  is said to be an *eigenvector* of an operator  $A$  if it generates a one-dimensional invariant subspace.

It is clear that this is the case if and only if there exists a number  $\lambda \in \mathbb{K}$  such that

$$(1) \quad A\mathbf{x} = \lambda\mathbf{x}.$$

Every number  $\lambda \in \mathbb{K}$  for which there exists a vector  $\mathbf{x} \neq 0$  that satisfies relation (1) (and hence is an eigenvector of the operator  $A$ ) is called an *eigenvalue* of the operator  $A$ . An eigenvector  $\mathbf{x}$  for which, for a given  $\lambda$ , (1) holds is said to *belong* to the eigenvalue  $\lambda$ .

It is convenient to assume that belonging to every eigenvalue  $\lambda$  is also a zero vector  $0$  (which is not by definition an eigenvector). Then for any eigenvalue  $\lambda$  the set  $\mathcal{P}_\lambda$  of all vectors  $\mathbf{x} \in \mathcal{V}$  belonging to it is obviously a subspace. It is called a *proper subspace belonging to the eigenvalue  $\lambda$* . Its dimension  $p_\lambda = \dim \mathcal{P}_\lambda$  is called the *geometric multiplicity* of the eigenvalue  $\lambda$ . By definition  $1 \leq p_\lambda \leq n$ .

For any eigenvector  $\mathbf{x} \neq 0$  belonging to an eigenvalue  $\lambda$  the one-dimensional invariant subspace  $[\mathbf{x}]$  it generates lies entirely in  $\mathcal{P}_\lambda$ . Conversely, each one-dimensional subspace of the space  $\mathcal{P}_\lambda$  is invariant and hence, in particular,

the space  $\mathcal{P}_\lambda$  is decomposable as a direct sum of one-dimensional invariant subspaces. To obtain such a decomposition it is sufficient to choose an arbitrary basis in  $\mathcal{P}_\lambda$ .

Geometrically the subspace  $\mathcal{P}_\lambda$  can be characterized as a maximum invariant subspace on which the operator  $A$  (more precisely, its restriction  $A|_{\mathcal{P}_\lambda}$ ) is a scalar operator  $\lambda E$ . One can also say that  $\mathcal{P}_\lambda$  is the kernel of the operator  $A - \lambda E$ :

$$\mathcal{P}_\lambda = \text{Ker} (A - \lambda E).$$

Indeed, the equation  $(A - \lambda E) \mathbf{x} = 0$  is exactly equivalent to equation (1).  $\square$

We thus see that a number  $\lambda \in \mathbb{K}$  is an eigenvalue of an operator  $A$  if and only if the operator  $A - \lambda E$  has a nonzero kernel, i.e. is noninvertible (singular); see Proposition 2 of the preceding lecture. In other words,  $\lambda$  is an eigenvalue if and only if

$$\det (A - \lambda E) = 0,$$

where  $A$  is the matrix of the operator  $A$  in an arbitrary basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .

The determinant

$$\det (A - \lambda E) = \begin{vmatrix} a_1^1 - \lambda & \dots & a_n^1 \\ \vdots & \ddots & \vdots \\ a_1^n & \dots & a_n^n - \lambda \end{vmatrix}$$

is, as is easily seen, a polynomial of degree  $n$  in  $\lambda$ . *This polynomial is independent of the choice of basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ .* Indeed, in any other basis the matrix of the operator  $A$  has the form  $C^{-1}AC$  (see formula (5) of the preceding lecture) and

$$C^{-1}AC - \lambda E = C^{-1} (A - \lambda E) C$$

and therefore

$$\begin{aligned} \det (C^{-1}AC - \lambda E) &= (\det C)^{-1} \det (A - \lambda E) (\det C) = \\ &= \det (A - \lambda E). \quad \square \end{aligned}$$

**Definition 2.** The polynomial

$$f_A(\lambda) = \det (A - \lambda E)$$

is called a *characteristic polynomial* of an operator  $A$  and its roots (in the corresponding extension over a field  $\mathbb{K}$ ) are called *characteristic roots* of the operator  $A$ .

According to what has been said above *any eigenvalue of the operator  $A$  is its characteristic root and conversely any characteristic root in the field  $\mathbb{K}$  is an eigenvalue.*  $\square$

A practical method for finding proper spaces is based on this statement (and on the fact that  $\mathcal{P}_\lambda = \text{Ker}(A - \lambda E)$ ). First, solving the equation  $f_A(\lambda) = 0$ , we find all its roots lying in  $\mathbb{K}$  and then find a subspace  $\mathcal{P}_{\lambda_i}$  for every such root  $\lambda_i$  by solving a system of homogeneous linear equations with matrix  $A - \lambda_i E$ .

The multiplicity of the eigenvalue  $\lambda_0$  as a root of a characteristic polynomial, i.e. a number  $n_{\lambda_0}$  such that the polynomial  $f_A(\lambda)$  is divisible by  $(\lambda - \lambda_0)^{n_{\lambda_0}}$  but is not by  $(\lambda - \lambda_0)^{n_{\lambda_0}+1}$ , is called the *algebraic multiplicity* of the eigenvalue  $\lambda_0$ . It is easy to see that the *algebraic multiplicity of an eigenvalue is at least as high as its geometric multiplicity*:

$$p_{\lambda_0} \leq n_{\lambda_0}.$$

Indeed, let  $p = p_{\lambda_0}$  and let  $e_1, \dots, e_n$  be a basis of a space  $\mathcal{V}$  such that  $\mathcal{P}_{\lambda_0} = [e_1, \dots, e_p]$ . In that basis the matrix of the operator  $A$  has the form

$$(2) \quad \begin{pmatrix} A_1 & C \\ 0 & B \end{pmatrix}$$

and hence

$$f_A(\lambda) = \det(A - \lambda E) = \det(A_1 - \lambda E) \cdot \det(B - \lambda E).$$

But  $A_1$  is the matrix of the operator

$$A|_{\mathcal{P}_{\lambda_0}} = \lambda_0 E$$

and hence  $\det(A_1 - \lambda E) = (\lambda_0 - \lambda)^p$ . This proves that the polynomial  $f_A(\lambda)$  is divisible by  $(\lambda - \lambda_0)^p$  and hence  $p \leq n_{\lambda_0}$ .  $\square$

**Remark.** The operator  $A$  has a matrix of the form (2) in any basis for which the subspace  $\mathcal{P} = [e_1, \dots, e_n]$  is invariant, with  $A_1$  the matrix of the operator  $A|_{\mathcal{P}}$  and



$B$  the matrix of an induced operator  $\mathbf{B}: \mathcal{V}/\mathcal{P} \rightarrow \mathcal{V}/\mathcal{P}$ . This proves that for any invariant subspace  $\mathcal{P} \subset \mathcal{V}$  there is a decomposition

$$f_{\mathbf{A}}(\lambda) = f_{\mathbf{A}|_{\mathcal{P}}}(\lambda) f_{\mathbf{B}}(\lambda).$$

Let  $\lambda_1, \dots, \lambda_m$  be *distinct* eigenvalues of the operator  $\mathbf{A}$  and let

$$\mathcal{P}_1 = \mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_m = \mathcal{P}_{\lambda_m}$$

be the proper subspaces belonging to them.

**Proposition 1.** *The sum*

$$\mathcal{P} = \mathcal{P}_1 + \dots + \mathcal{P}_m$$

*of subspaces  $\mathcal{P}_1, \dots, \mathcal{P}_m$  is a direct sum, i.e. the equation*

$$(3) \quad \mathbf{x}_1 + \dots + \mathbf{x}_m = 0,$$

*where  $\mathbf{x}_1 \in \mathcal{P}_1, \dots, \mathbf{x}_m \in \mathcal{P}_m$ , holds if and only if*

$$\mathbf{x}_1 = 0, \dots, \mathbf{x}_m = 0.$$

*Proof.* We proceed by induction on  $m$ . For  $m = 1$  the statement is obvious (and meaningless). Suppose we have already proved that the sum of  $m - 1$  spaces  $\mathcal{P}_1, \dots, \mathcal{P}_{m-1}$  is direct. By applying to (3) the operator  $\mathbf{A}$  we obtain the relation

$$(4) \quad \lambda_1 \mathbf{x}_1 + \dots + \lambda_m \mathbf{x}_m = 0.$$

On multiplying (3) by  $\lambda_m$  and subtracting from (4) we then get

$$(\lambda_1 - \lambda_m) \mathbf{x}_1 + \dots + (\lambda_{m-1} - \lambda_m) \mathbf{x}_{m-1} = 0.$$

By induction hypothesis it follows that

$$(\lambda_1 - \lambda_m) \mathbf{x}_1 = 0, \dots, (\lambda_{m-1} - \lambda_m) \mathbf{x}_{m-1} = 0$$

and hence (since under the hypothesis  $\lambda_1 - \lambda_m \neq 0, \dots, \lambda_{m-1} - \lambda_m \neq 0$ ) that

$$\mathbf{x}_1 = 0, \dots, \mathbf{x}_{m-1} = 0.$$

But then, according to (3), also  $\mathbf{x}_m = 0$ .  $\square$

Let there exist (distinct) eigenvalues

$$(5) \quad \lambda_1, \dots, \lambda_m,$$

such that

$$(6) \quad \mathcal{P}_{\lambda_1} \oplus \dots \oplus \mathcal{P}_{\lambda_m} = \mathcal{V}$$

and hence

$$(7) \quad p_{\lambda_1} + \dots + p_{\lambda_m} = n.$$

It is easy to see that *numbers (5) exhaust all the eigenvalues of the operator A*. Indeed, for any other eigenvalue  $\lambda_0$  the subspace  $\mathcal{P}_{\lambda_0}$  would form with  $\mathcal{V}$ , according to Proposition 1, a direct sum, which is impossible.  $\square$

On choosing a basis in each of the spaces  $\mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_{\lambda_m}$  we obtain a basis of a space  $\mathcal{V}$  consisting of eigenvectors. The matrix of the operator A in that basis is diagonal:

$$(8) \quad \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

and its diagonal elements are the eigenvalues (5), each  $\lambda_i$  repeated  $p_{\lambda_i}$  times.

Conversely, let there exist in a space  $\mathcal{V}$  a basis in which the matrix A of the operator A is diagonal. Then the vectors of the basis are eigenvectors and the diagonal elements of the matrix A are the eigenvalues of the operator A. Let  $\lambda_1, \dots, \lambda_m$  be all distinct diagonal elements of the matrix A and let the element  $\lambda_i$ ,  $i = 1, \dots, m$ , be repeated  $q_i$  times. Also let  $\mathcal{Q}_i$ ,  $i = 1, \dots, m$ , be a subspace of the space  $\mathcal{V}$  generated by the vectors of the basis belonging to the eigenvalue  $\lambda_i$ . Then  $\dim \mathcal{Q}_i = q_i$ ,

$$\mathcal{Q}_1 \oplus \dots \oplus \mathcal{Q}_m = \mathcal{V}$$

and  $\mathcal{Q}_i \subset \mathcal{P}_{\lambda_i}$ . Therefore, in particular,

$$(9) \quad q_1 + \dots + q_m = n \text{ and } q_1 \leq p_{\lambda_1}, \dots, q_m \leq p_{\lambda_m}.$$

But according to Proposition 1 the sum  $\mathcal{P}_{\lambda_1} + \dots + \mathcal{P}_{\lambda_m}$  of the subspaces  $\mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_{\lambda_m}$  is their direct sum and hence has dimension  $p_{\lambda_1} + \dots + p_{\lambda_m}$ . Therefore  $p_{\lambda_1} + \dots$

$\dots + p_{\lambda_m} \leq n$ , whence by virtue of relations (9) it follows that

$$q_1 = p_{\lambda_1}, \dots, q_m = p_{\lambda_m},$$

i.e. that

$$\mathcal{Q}_1 = \mathcal{P}_{\lambda_1}, \dots, \mathcal{Q}_m = \mathcal{P}_{\lambda_m}.$$

Consequently, for the subspaces  $\mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_{\lambda_m}$  decomposition (6) holds.

Since the existence of a basis in which the matrix  $A$  of the operator  $A$  is diagonal is equivalent to the decomposability of the space  $\mathcal{V}$  as a direct sum of one-dimensional invariant subspaces, this proves the following proposition.

**Proposition 2.** *For any linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  the following statements are equivalent:*

1° *There exist eigenvalues  $\lambda_1, \dots, \lambda_m$  such that*

$$\mathcal{P}_{\lambda_1} \oplus \dots \oplus \mathcal{P}_{\lambda_m} = \mathcal{V}.$$

2° *The space  $\mathcal{V}$  is a direct sum of one-dimensional subspaces invariant under the operator  $A$ .*

3° *In the space  $\mathcal{V}$  there exists a basis consisting of eigenvectors, i.e. a basis in which the matrix of the operator  $A$  is diagonal.*  $\square$

The eigenvalues  $\lambda_1, \dots, \lambda_m$  appearing in 1° (and implicitly in 2°) exhaust all the eigenvalues of the operator  $A$ . Every basis in which the matrix of the operator  $A$  is diagonal is obtained by combining the bases of the spaces  $\mathcal{P}_{\lambda_1}, \dots, \dots, \mathcal{P}_{\lambda_m}$ , so that for any eigenvalue  $\lambda_i$  in that basis there are exactly  $p_{\lambda_i}$  vectors belonging to  $\lambda_i$ .

**Definition 3.** An operator  $A$  is said to be *diagonalizable* if Statements 1° to 3° hold for it.

Computing the characteristic polynomial of the diagonalizable operator  $A$  in the basis consisting of eigenvectors we get immediately

$$f_A(\lambda) = (\lambda - \lambda_1)^{p_1} \dots (\lambda - \lambda_m)^{p_m},$$

where  $\lambda_1, \dots, \lambda_m$  are the eigenvalues of the operator  $A$  and  $p_1 = p_{\lambda_1}, \dots, p_m = p_{\lambda_m}$  are their geometric multiplicities. This proves that for a diagonalizable operator any

of its characteristic roots  $\lambda_0$  is in the field  $\mathbb{K}$  (and hence is an eigenvalue) and that its algebraic multiplicity  $n_{\lambda_0}$  coincides with its geometric multiplicity  $p_{\lambda_0}$ .  $\square$

It turns out that this necessary condition for diagonalizability is a sufficient condition as well, so that the following theorem holds.

**Theorem 1.** *A linear operator  $A$  is diagonalizable if and only if any of its characteristic roots  $\lambda_0$  is in the field  $\mathbb{K}$  and  $n_{\lambda_0} = p_{\lambda_0}$ .*

*Proof.* It is necessary for us to prove only the sufficiency of this condition.

Let  $\lambda_1, \dots, \lambda_m$  be all characteristic roots of the operator  $A$ . By the hypothesis they are in  $\mathbb{K}$  and hence are also eigenvalues. Therefore subspaces  $\mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_{\lambda_m}$  are defined the dimension of whose sum (a direct one as we know) is

$$p_{\lambda_1} + \dots + p_{\lambda_m} = n_{\lambda_1} + \dots + n_{\lambda_m} = n$$

(the sum of the multiplicities of all roots of a polynomial is equal to its degree). Hence  $\mathcal{P}_{\lambda_1} \oplus \dots \oplus \mathcal{P}_{\lambda_m} = \mathcal{V}$ .  $\square$

**Definition 4.** The set of all characteristic roots of an operator  $A$  is called the *spectrum* of the operator. The spectrum is said to be *simple* if every characteristic root  $\lambda_0$  is a simple root of the characteristic polynomial, i.e. if  $n_{\lambda_0} = 1$ .

The *spectrum* is said to lie in  $\mathbb{K}$  if all characteristic roots lie in  $\mathbb{K}$ .

**Proposition 3.** *Any operator with simple spectrum in  $\mathbb{K}$  is diagonalizable.*

*Proof.* Since  $1 \leq p_{\lambda} \leq n_{\lambda}$ , for  $n_{\lambda} = 1$  we necessarily have  $p_{\lambda} = 1$  (and hence  $p_{\lambda} = n_{\lambda}$ ).  $\square$

This diagonalizability condition is not necessary, but it is convenient for a practical check.

Let  $\mathcal{P}$  be an arbitrary invariant (under an operator  $A$ ) subspace of a space  $\mathcal{V}$ . Since (see the remark above) the characteristic polynomial  $f_B(\lambda)$  of the induced operator  $B: \mathcal{V}/\mathcal{P} \rightarrow \mathcal{V}/\mathcal{P}$  divides the characteristic polynomial  $f_A(\lambda)$  of the operator  $A$ , each characteristic root of the operator  $B$  is a characteristic root of the operator  $A$  of at least the same algebraic multiplicity. In particular, if the spectrum of

the operator  $A$  lies in  $\mathbb{K}$ , so does the spectrum of the operator  $B$  and hence there exists at least one eigenvalue  $\lambda_0$  for  $B$ . Let  $\mathbf{x}_0 + \mathcal{F}$  be the corresponding eigenvector of the operator  $B$ . The equation  $B(\mathbf{x}_0 + \mathcal{F}) = \lambda_0(\mathbf{x}_0 + \mathcal{F})$  implies that  $A\mathbf{x}_0 = \lambda_0\mathbf{x}_0 + \mathbf{a}_0$ , where  $\mathbf{a}_0 \in \mathcal{F}$ , from which it follows that the subspace  $\mathcal{Q}$  generated by the subspace  $\mathcal{F}$  and vector  $\mathbf{x}_0$  (i.e. consisting of all vectors of the form  $k\mathbf{x}_0 + \mathbf{a}$ , where  $k \in \mathbb{K}$  and  $\mathbf{a} \in \mathcal{F}$ ; note that  $\mathbf{x}_0 \notin \mathcal{F}$ ) is invariant under  $A$ . Since  $\dim \mathcal{Q} = \dim \mathcal{F} + 1$ , this proves the following proposition.

**Proposition 4.** *If the spectrum of a linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  lies in  $\mathbb{K}$ , then any of its invariant subspaces is contained in an invariant subspace of dimension higher by unity.  $\square$*

Consequently, beginning with the subspace  $\mathcal{F}_0 = 0$ , we can construct an ascending chain of invariant subspaces

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_n = \mathcal{V}$$

of dimensions  $0, 1, \dots, n$ . It is clear that in the corresponding basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of the space  $\mathcal{V}$ , i.e. in a basis such that  $\mathcal{F}_i = [\mathbf{e}_1, \dots, \mathbf{e}_i]$  for any  $i = 1, \dots, n$ , the matrix of the operator  $A$  is a triangular matrix

$$(10) \quad \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_m \end{pmatrix}$$

whose diagonal elements are the eigenvalues of the operator  $A$ , each repeated as many times as is its multiplicity. This proves the following proposition.

**Proposition 5.** *For any linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  with spectrum in  $\mathbb{K}$  there is in the space  $\mathcal{V}$  a basis in which the matrix of the operator is triangular.*

When  $\mathbb{K} = \mathbb{C}$  this corollary applies of course to any linear operator.

We shall first obtain a more precise result for operators of one special class.

**Definition 5.** An operator  $A$  (matrix  $A$ ) is said to be *nilpotent* if there exists a natural number  $m$  such that  $A^m = 0$  (respectively  $A^m = 0$ ). The smallest of such  $m$  is called the *degree of nilpotency* of the operator (the matrix).

It is easy to see that *all eigenvalues of a nilpotent operator are equal to zero*. Indeed, if  $A\mathbf{x} = \lambda\mathbf{x}$ , then  $A^k\mathbf{x} = \lambda^k\mathbf{x}$  for any  $k$  and hence when  $A^m = 0$  and  $\mathbf{x} \neq 0$ , necessarily  $\lambda^m = 0$ , i.e.  $\lambda = 0$ .  $\square$

Therefore it is impossible for a nonzero nilpotent operator to be diagonalizable.

One example of a nilpotent operator is an operator for which there exists a vector  $\mathbf{e} \neq 0$  such that the vectors  $\mathbf{e}, A\mathbf{e}, \dots, A^{n-1}\mathbf{e}$

constitute a basis of a space  $\mathcal{V}$ , and  $A^n\mathbf{e} = 0$ . In the basis

$$\mathbf{e}_1 = A^{n-1}\mathbf{e}, \dots, \mathbf{e}_{n-1} = A\mathbf{e}, \mathbf{e}_n = \mathbf{e}$$

the matrix of this operator is

$$(11) \quad \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & \ddots \\ 0 & & & 1 \\ & & & & 0 \end{pmatrix}.$$

Operators of such a form are called *cyclic operators*.

For any vector  $\mathbf{x} = x^1\mathbf{e}_1 + \dots + x^n\mathbf{e}_n$  and any  $m \leq n$  we have  $A^m\mathbf{x} = x^{m+1}\mathbf{e}_1 + \dots + x^n\mathbf{e}_{n-m}$  and, in particular,  $A^n\mathbf{x} = 0$ . Thus a *cyclic operator is nilpotent and its degree of nilpotency equals  $n$* .  $\square$

When  $n = 1$  the cyclic operator is zero.

It turns out that an arbitrary nilpotent operator reduces to cyclic operators.

**Proposition 6.** *For any nilpotent operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  there exists a decomposition*

$$\mathcal{V} = \mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_m$$

*of the space  $\mathcal{V}$  as a direct sum of invariant subspaces on each of which the operator  $A$  induces a cyclic operator.*

We shall prove this proposition in the next lecture.

# Lecture 16

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*Decomposition of a nilpotent operator as a direct sum of cyclic operators. Root subspaces. Normal Jordan form. The Hamilton-Cayley theorem*

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Let  $A: \mathcal{V} \rightarrow \mathcal{V}$  be an arbitrary nilpotent operator and let

$$\mathcal{P}_i = \text{Im } A^i, \quad i = 0, 1, \dots, m,$$

where  $m$  is the degree of nilpotency of the operator  $A$ . Since  $A^{i+1}\mathbf{x} = A^i(A\mathbf{x})$ , we have

$$0 = \mathcal{P}_m \subset \mathcal{P}_{m-1} \subset \dots \subset \mathcal{P}_{i+1} \subset \mathcal{P}_i \subset \dots \subset \mathcal{P}_1 \subset \mathcal{P}_0 = \mathcal{V}.$$

( $A^0 = E$  by definition, and hence  $\mathcal{P}_0 = \mathcal{V}$  even for  $A = 0$ ).

By construction  $A(\mathcal{P}_i) = \mathcal{P}_{i+1}$ ,  $0 \leq i < m$ , from which it follows in particular that  $\mathcal{P}_{m-1} \subset \text{Ker } A$ . Hence for any basis

$$(1) \quad \mathbf{e}_1^{(m-1)}, \dots, \mathbf{e}_{p_{m-1}}^{(m-1)}, \quad p_{m-1} = \dim \mathcal{P}_{m-1}$$

of the space  $\mathcal{P}_{m-1}$  the relations

$$(2) \quad A\mathbf{e}_1^{(m-1)} = 0, \dots, A\mathbf{e}_{p_{m-1}}^{(m-1)} = 0$$

hold. In addition there are in the space  $\mathcal{P}_{m-2}$  vectors

$$\mathbf{e}_1^{(m-2)}, \dots, \mathbf{e}_{p_{m-1}}^{(m-2)}$$

such that

$$(3) \quad A\mathbf{e}_1^{(m-2)} = \mathbf{e}_1^{(m-1)}, \dots, A\mathbf{e}_{p_{m-1}}^{(m-2)} = \mathbf{e}_{p_{m-1}}^{(m-1)}.$$

It turns out that the *vectors*

$$(4) \quad \begin{array}{c} \mathbf{e}_1^{(m-1)}, \dots, \mathbf{e}_{p_{m-1}}^{(m-1)} \\ \mathbf{e}_1^{(m-2)}, \dots, \mathbf{e}_{p_{m-1}}^{(m-2)} \end{array}$$

of the space  $\mathcal{F}_{m-2}$  are linearly independent. Indeed, if

$$k_1 \mathbf{e}_1^{(m-1)} + \dots + k_p \mathbf{e}_p^{(m-1)} + l_1 \mathbf{e}_1^{(m-2)} + \dots + l_p \mathbf{e}_p^{(m-2)} = 0,$$

where  $p = p_{m-1}$ , then by applying to this equation the operator  $A$  we obtain by virtue of (2) and (3) the equation

$$l_1 \mathbf{e}_1^{(m-1)} + \dots + l_p \mathbf{e}_p^{(m-1)} = 0,$$

which is true (since (1) is a basis of the subspace  $\mathcal{F}_{m-1}$ ) only when  $l_1 = 0, \dots, l_p = 0$ . But then

$$k_1 \mathbf{e}_1^{(m-1)} + \dots + k_p \mathbf{e}_p^{(m-1)} = 0$$

and hence for the same reasons  $k_1 = 0, \dots, k_p = 0$ .  $\square$

Therefore we can extend the vectors (4) to some basis

$$(5) \quad \begin{array}{c} \mathbf{e}_1^{(m-1)}, \dots, \mathbf{e}_{p_{m-1}}^{(m-1)}, \\ \mathbf{e}_1^{(m-2)}, \dots, \mathbf{e}_{p_{m-1}}^{(m-2)}, \dots, \mathbf{e}_{p_{m-2}}^{(m-2)}, \end{array}$$

$$p_{m-2} = \dim \mathcal{F}_{m-2} - \dim \mathcal{F}_{m-1},$$

of the space  $\mathcal{F}_{m-2}$ . It is easy to see that the complementary vectors

$$(6) \quad \mathbf{e}_{p_{m-1}+1}^{(m-2)}, \dots, \mathbf{e}_{p_{m-2}}^{(m-2)}$$

can be taken from the kernel  $\text{Ker } A$  of the operator  $A$ , i.e. so that we have

$$(7) \quad A \mathbf{e}_{p_{m-1}+1}^{(m-2)} = 0, \dots, A \mathbf{e}_{p_{m-2}}^{(m-2)} = 0.$$

Indeed, since the vectors (1) constitute a basis of the space  $\mathcal{F}_{m-1} = A(\mathcal{F}_{m-2})$ , we have for an arbitrary choice of vectors (6) and any  $i = 1, \dots, p_{m-2} - p_{m-1}$ ,

$$A \mathbf{e}_{p+i}^{(m-2)} = x_i^1 \mathbf{e}_1^{(m-1)} + \dots + x_i^p \mathbf{e}_p^{(m-1)},$$

where

$$p = p_{m-1}.$$



Therefore, by replacing the vectors  $e_{p+1}^{(m-2)}$  with the vectors

$$e_{p+i}^{(m-2)} = x_i^1 e_1^{(m-2)} = \dots = x_i^p e_p^{(m-2)}$$

we satisfy conditions (7).  $\square$

Since  $A(\mathcal{F}_{m-3}) = \mathcal{F}_{m-2}$ , there exist in the subspace  $\mathcal{F}_{m-3}$  vectors

$$(8) \quad e_1^{(m-3)}, \dots, e_{p_{m-2}}^{(m-3)},$$

such that

$$(9) \quad Ae_1^{(m-3)} = e_1^{(m-2)}, \dots, Ae_{p_{m-2}}^{(m-3)} = e_{p_{m-2}}^{(m-2)}.$$

It can be shown by the same method as that used for the family of vectors (4) that the *vectors (5) and (8) constitute together a linearly independent family*. Indeed, by applying the operator  $A$  to an arbitrary linear combination of these vectors we obtain by virtue of (2), (3), (7), and (9) a linear combination of vectors (5). The corresponding coefficients are therefore zero and hence all that is left of the entire combination is a combination of vectors (1) and (7) from the kernel. Since these vectors are linearly independent, the remaining coefficients are also zero.  $\square$

This linearly independent family can be extended to the basis

$$\begin{aligned} &e_1^{(m-1)}, \dots, e_{p_{m-1}}^{(m-1)}, \\ &e_1^{(m-2)}, \dots, e_{p_{m-1}}^{(m-2)}, \dots, e_{p_{m-2}}^{(m-2)}, \\ &e_1^{(m-3)}, \dots, e_{p_{m-1}}^{(m-3)}, \dots, e_{p_{m-2}}^{(m-3)}, \dots, e_{p_{m-3}}^{(m-3)}, \end{aligned}$$

the argument employed for the vectors (6) similarly showing that the complementary vectors

$$e_{p_{m-2}+1}^{(m-3)}, \dots, e_{p_{m-3}}^{(m-3)}$$

can be taken from the kernel of the operator  $A$ , i.e. so that we have

$$Ae_{p_{m-2}+1}^{(m-3)} = 0, \dots, Ae_{p_{m-3}}^{(m-3)} = 0.$$

Continuing this construction step by step we finally obtain in the space  $\mathcal{F}_0 = \mathcal{V}$  a basis whose vectors are arranged in a stepped array of the form

$$\begin{array}{ccccccc} e_1^{(m-1)}, & \dots, & e_{p_{m-1}}^{(m-1)}, & & & & \\ e_1^{(m-2)}, & \dots, & e_{p_{m-1}}^{(m-2)}, & \dots, & e_{p_{m-2}}^{(m-2)}, & & \\ e_1^{(m-3)}, & \dots, & e_{p_{m-1}}^{(m-3)}, & \dots, & e_{p_{m-2}}^{(m-3)}, & \dots, & e_{p_{m-3}}^{(m-3)}, \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ e_1^{(0)}, & \dots, & e_{p_{m-1}}^{(0)}, & \dots, & e_{p_{m-2}}^{(0)}, & \dots, & e_{p_{m-3}}^{(0)}, \dots, e_{p_0}^{(0)} \end{array}$$

having the property that under the operator  $A$  the vectors of each column mount a step while remaining in the same column (the uppermost vectors becoming zero).

This property means by definition that the vectors of each column generate an invariant subspace, the restriction of the operator  $A$  to the subspace being a cyclic operator (the lowest vector in the column obviously serves as a vector  $e$  for that operator—see the preceding lecture). Since the space  $\mathcal{V}$  is a direct sum of these invariant subspaces, Proposition 6 of the preceding lecture is thus completely proved.  $\square$

We now return to arbitrary operators.

The proper subspace  $\mathcal{F}_\lambda$  can be defined as a maximum subspace of a space  $\mathcal{V}$  on which the operator  $A - \lambda E$  is equal to zero. By analogy we introduce the following definition.

**Definition 1.** A maximum invariant subspace  $\mathcal{R}_\lambda$  of a space  $\mathcal{V}$  on which the operator  $A - \lambda E$  is nilpotent is called a *root subspace* of the operator  $A$  belonging to an eigenvalue  $\lambda$ .

We explain this definition.

A vector  $x \neq 0$  of a space  $\mathcal{V}$  is said to be a *root vector* belonging to an eigenvalue  $\lambda$  if there exists an integer  $m \geq 0$  such that

$$(A - \lambda E)^m x = 0.$$

It is clear that for any  $k \in \mathbb{K}$  the vector  $kx$  is also a root vector belonging to  $\lambda$  (or zero). It is easy to see as well that a similar statement is also true for a sum of root vectors

belonging to the same eigenvalue, for if  $(A - \lambda E)^{m_1} \mathbf{x}_1 = 0$  and  $(A - \lambda E)^{m_2} \mathbf{x}_2 = 0$ , then  $(A - \lambda E)^m (\mathbf{x}_1 + \mathbf{x}_2) = 0$ , where  $m = \max(m_1, m_2)$ . This means that the zero vector-supplemented set of all root vectors belonging to a given eigenvalue  $\lambda$  is a subspace of the space  $\mathcal{V}$ . This subspace is exactly the subspace  $\mathcal{R}_\lambda$  described in Definition 1, since it is obviously invariant under the operator  $A - \lambda E$  and hence under the operator  $A$ .

It is clear that any subspace  $\mathcal{R}$  of a space  $\mathcal{V}$  invariant under an operator  $A$  is also invariant under every operator of the form  $A - \mu E$ ,  $\mu \in \mathbb{K}$ . In particular the subspace  $\mathcal{R}_\lambda$  is invariant under any operator of the form  $A - \mu E$ ,  $\mu \in \mathbb{K}$  and hence the operator

$$(10) \quad (A - \mu E) | \mathcal{R}_\lambda$$

is defined.

**Proposition 1.** *When  $\mu \neq \lambda$  the operator (10) is invertible.*

*Proof.* If

$$(A - \mu E) \mathbf{x} = 0,$$

where  $\mathbf{x} \in \mathcal{R}_\lambda$ , then  $A\mathbf{x} = \lambda\mathbf{x}$  and therefore  $(A - \mu E) \mathbf{x} = (\lambda - \mu) \mathbf{x}$ . Hence either the vector  $\mathbf{x}$  is zero or the number  $\lambda - \mu$  is an eigenvalue of the nilpotent operator

$$(A - \lambda E) | \mathcal{R}_\lambda.$$

But, as we know, all eigenvalues of an arbitrary nilpotent operator are zero. Since by the hypothesis  $\mu \neq \lambda$ , it follows that  $\mathbf{x} = 0$ . This proves that the kernel of the operator (10) contains only a zero vector. Therefore (Proposition 2 of Lecture 14) the operator (10) is invertible.  $\square$

Now we are in a position to prove the analogue of Proposition 1 of the preceding lecture for root spaces.

Let  $\lambda_1, \dots, \lambda_m$  be distinct eigenvalues of an operator  $A$  and let

$$\mathcal{R}_1 = \mathcal{R}_{\lambda_1}, \dots, \mathcal{R}_m = \mathcal{R}_{\lambda_m}$$

be root subspaces belonging to them.

**Proposition 2.** *A sum*

$$\mathcal{R}_1 + \dots + \mathcal{R}_m$$

of subspaces  $\mathcal{R}_1, \dots, \mathcal{R}_m$  is direct, i.e. the equation

$$\mathbf{x}_1 + \dots + \mathbf{x}_m = \mathbf{0},$$

where  $\mathbf{x}_1 \in \mathcal{R}_1, \dots, \mathbf{x}_m \in \mathcal{R}_m$  holds if and only if

$$\mathbf{x}_1 = \mathbf{0}, \dots, \mathbf{x}_m = \mathbf{0}.$$

*Proof* (cf. the proof of Proposition 1 in the preceding lecture). For  $m = 1$  the proposition is obvious. Suppose it is already proved for  $m - 1$  root subspaces. Since  $\mathbf{x}_m \in \mathcal{R}_m$ , there exists a number  $s$  such that

$$(\mathbf{A} - \lambda_m \mathbf{E})^s \mathbf{x} = \mathbf{0}.$$

Therefore

$$\mathbf{y}_1 + \dots + \mathbf{y}_{m-1} = \mathbf{0},$$

where

$$\mathbf{y}_1 = (\mathbf{A} - \lambda_m \mathbf{E})^s \mathbf{x}_1, \dots, \mathbf{y}_{m-1} = (\mathbf{A} - \lambda_m \mathbf{E})^s \mathbf{x}_{m-1}.$$

Since the subspaces  $\mathcal{R}_1, \dots, \mathcal{R}_{m-1}$  are invariant under the operator  $\mathbf{A} - \lambda_m \mathbf{E}$ , we have  $\mathbf{y}_1 \in \mathcal{R}_1, \dots, \mathbf{y}_{m-1} \in \mathcal{R}_{m-1}$  and hence by induction hypothesis

$$\mathbf{y}_1 = \mathbf{0}, \dots, \mathbf{y}_{m-1} = \mathbf{0}.$$

Since according to Proposition 1 the operator  $\mathbf{A} - \lambda_m \mathbf{E}$  on the subspaces  $\mathcal{R}_1, \dots, \mathcal{R}_{m-1}$  is invertible, it follows that

$$\mathbf{x}_1 = \mathbf{0}, \dots, \mathbf{x}_{m-1} = \mathbf{0}$$

and hence  $\mathbf{x}_m = \mathbf{0}$ .  $\square$

An advantage of root subspaces over proper subspaces manifests itself in the following proposition.

**Proposition 3.** *For any eigenvalue  $\lambda$  of an operator  $\mathbf{A}$  the dimension of a root subspace  $\mathcal{R}_\lambda$  is equal to the algebraic multiplicity of the eigenvalue:*

$$\dim \mathcal{R}_\lambda = n_\lambda.$$

*Proof.* Let  $\mathbf{A}_1 = \mathbf{A} |_{\mathcal{R}_\lambda}$  and let  $\mathbf{B}$  be an operator  $\mathcal{V} / \mathcal{R}_\lambda \rightarrow \mathcal{V} / \mathcal{R}_\lambda$  induced by the operator  $\mathbf{A}$ . Then  $f_{\mathbf{A}} = f_{\mathbf{A}_1} f_{\mathbf{B}}$ . Consequently, if  $\dim \mathcal{R}_\lambda < n_\lambda$ , then the number  $\lambda$  is a root of the polynomial  $f_{\mathbf{B}}$ . Hence, since  $\lambda \in \mathbb{K}$ , it is at the same time an eigenvalue of the operator  $\mathbf{B}$ .

Let  $\mathbf{x}_0 \in \mathcal{R}_\lambda$  be the corresponding eigenvector. Then

$$A\mathbf{x}_0 = \lambda\mathbf{x}_0 + \mathbf{a}_0,$$

where  $\mathbf{a}_0 \in \mathcal{R}_\lambda$ ,  $\mathbf{a}_0 = (A - \lambda E)\mathbf{x}_0$ . Therefore there exists  $m$  such that  $(A - \lambda E)^m \mathbf{a}_0 = 0$ . But then

$$(A - \lambda E)^{m+1} \mathbf{x}_0 = 0$$

and hence  $\mathbf{x}_0 \in \mathcal{R}_\lambda$ , which is impossible. The obtained contradiction shows that  $\dim \mathcal{R}_\lambda = n_\lambda$ .  $\square$

It follows from Proposition 3 that if the spectrum of the operator  $A$  is in  $\mathbb{K}$  and  $\lambda_1, \dots, \lambda_m$  are all the eigenvalues (characteristic roots) it has, then

$$\dim (\mathcal{R}_{\lambda_1} \oplus \dots \oplus \mathcal{R}_{\lambda_m}) = n_{\lambda_1} + \dots + n_{\lambda_m} = n$$

and hence

$$\mathcal{R}_{\lambda_1} \oplus \dots \oplus \mathcal{R}_{\lambda_m} = \mathcal{V}.$$

Thus the following theorem holds.

**Theorem 1.** *For any linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$ , whose spectrum lies in  $\mathbb{K}$  the space  $\mathcal{V}$  is a direct sum of the root subspaces of that operator:*

$$(11) \quad \mathcal{V} = \mathcal{R}_{\lambda_1} \oplus \dots \oplus \mathcal{R}_{\lambda_m}. \quad \square$$

To say that an invariant subspace  $\mathcal{R}$  of a space  $\mathcal{V}$  is a root subspace  $\mathcal{R}_\lambda$  is the same as to say that the restriction of an operator  $A$  to that subspace is a sum  $\lambda E + B$  of a scalar operator  $\lambda E$  and some nilpotent operator  $B$ . But according to Proposition 5 of the preceding lecture, for an operator  $B$  there exists a decomposition of a subspace  $\mathcal{R}$  as a direct sum of subspaces invariant (under  $B$  and hence under  $A$ ) on each of which the operator  $B$  induces a cyclic operator. On carrying out this decomposition for any root subspace of (11), we obtain a decomposition of a space  $\mathcal{V}$  as a direct sum of invariant subspaces on each of which the operator  $A$  induces an operator of the form

$$(12) \quad \lambda E + C,$$

where  $\lambda \in \mathbb{K}$  and  $C$  is some cyclic operator.

**Definition 2.** A matrix of the form

$$(13) \quad \begin{pmatrix} \lambda & 1 & 0 & \dots & 0 \\ 0 & \lambda & 1 & 0 & \dots & 0 \\ 0 & 0 & \lambda & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \dots & \lambda & 1 \\ 0 & 0 & \dots & \dots & \dots & \lambda \end{pmatrix}$$

is called a *Jordan cell*. A matrix  $A$  is said to have a *normal Jordan form* if it is a direct sum of Jordan cells (each in general with a different  $\lambda$ ).

Since the matrix of any cyclic operator has in an appropriate basis the form (11) from the preceding lecture, the matrix of the operator (12) in the same basis is the Jordan cell (13). By combining all the bases of the corresponding subspaces, therefore, we obtain a basis of the space  $\mathcal{V}$  in which the matrix of the operator  $A$  has a Jordan form. This proves the following theorem.

**Theorem 2 (reduction to Jordan form).** *For any linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  whose spectrum lies in  $\mathbb{K}$ , there exists a basis of the space  $\mathcal{V}$  in which its matrix has a normal Jordan form.  $\square$*

It turns out that up to the sequence of cells the *normal Jordan form of the matrix of the operator is uniquely determined*, i.e. the number of Jordan cells, their size and the corresponding numbers  $\lambda$  are the same for all bases in which the matrix of the operator has a normal form. As regards numbers  $\lambda$  this is obvious (since they are the eigenvalues of the operator). The statement concerning the number and size of Jordan cells we shall not prove.

**Remark.** The uniqueness of the normal Jordan form of a matrix follows immediately from the fact that the number of Jordan cells of order  $k$  corresponding to an eigenvalue  $\lambda$  is expressed by the formula

$$r(A - \lambda E)^{k-1} - 2r(A - \lambda E)^k + r(A - \lambda E)^{k+1},$$

where  $rC$  is the rank of a matrix  $C$ .

We leave the proof of the formula to the reader as a useful exercise.

We stress that when  $\mathbb{K} = \mathbb{C}$  the condition on the spectrum of an operator  $A$  in Theorem 2 holds for any operators, so that *over the field  $\mathbb{C}$  every linear operator reduces to Jordan form.*  $\square$

Here is one example of applying the results obtained.

Let

$$f(x) = a_0x^m + a_1x^{m-1} + \dots + a_m$$

be an arbitrary polynomial over a field  $\mathbb{K}$ . Then for any operator  $A$  (any matrix  $A$ ) the operator

$$f(A) = a_0A^m + a_1A^{m-1} + \dots + a_mE$$

(the matrix  $f(A) = a_0A^m + a_1A^{m-1} + \dots + a_mE$ ) called a *polynomial of the operator  $A$*  (of the matrix  $A$ ) is defined.

It is obvious that any subspace  $\mathcal{P} \subset \mathcal{V}$  invariant under the operator  $A$  is also invariant under every operator  $f(A)$ . Moreover

$$(14) \quad f(A)|_{\mathcal{P}} = f(A|_{\mathcal{P}}).$$

In particular, for any operator  $A$  an operator

$$f_A(A)$$

is defined, where  $f_A(\lambda) = \det(A - \lambda E)$  is the characteristic polynomial of the operator  $A$ . Let us compute this operator.

First let

$$(15) \quad A = \lambda_0 E + C,$$

where  $C$  is a cyclic operator. Then  $f_A(\lambda) = (\lambda_0 - \lambda)^n$  and  $C^n = 0$ . Therefore

$$f_A(A) = (\lambda_0 E - A)^n = (-C)^n = 0.$$

Now let the operator  $A$  (with a spectrum in  $\mathbb{K}$ ) be arbitrary and let

$$\mathcal{V} = \mathcal{P}_1 \oplus \dots \oplus \mathcal{P}_N$$

be a decomposition of the space  $\mathcal{V}$  as a direct sum of invariant subspaces on each of which the restriction  $A_i = A|_{\mathcal{P}_i}$

of the operator  $A$  has the form (15). Then, according to what has been proved,

$$(16) \quad f_{A_i}(A_i) = 0.$$

But, as we know, every polynomial  $f_{A_i}$  divides the polynomial  $f_A$  (moreover, the polynomial  $f_A$  is easily seen to be a product of polynomials  $f_{A_1}, \dots, f_{A_N}$ ). It therefore follows from (16) that

$$f_A(A_i) = 0.$$

Hence (see formula (14))

$$f_A(A) | \mathcal{P}_i = 0, \quad i = 1, \dots, N.$$

Thus the operator  $f_A(A)$  has the property that for any  $i = 1, \dots, N$  its restriction to a subspace  $\mathcal{P}_i$  is zero. Consequently this operator is equal to zero on the sum of these subspaces as well, i.e. on the entire space  $\mathcal{V}$ .

This proves the following theorem.

**Theorem 3 (Hamilton-Cayley theorem).** *Every operator annihilates its characteristic polynomial:*

$$f_A(A) = 0. \quad \square$$

We have proved the theorem for operators whose spectrum is in  $\mathbb{K}$  and thereby, in particular, for any operators over the field  $\mathbb{C}$ . In the next lecture we shall prove it (over the field  $\mathbb{R}$ ) for operators with an arbitrary spectrum.



# Lecture 17

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*Complexification of a linear operator • Proper subspaces belonging to characteristic roots • Operators whose complexification is diagonalizable*

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The results of the preceding lecture were obtained on the assumption that the spectrum of an operator  $A$  lies in the ground field  $\mathbb{K}$ . This condition is automatically fulfilled when  $\mathbb{K} = \mathbb{C}$ , but already for  $\mathbb{K} = \mathbb{R}$  it substantially restricts the applicability of the results. In this lecture we shall find out what results for operators failing to satisfy the condition. For simplicity we shall consider only the geometrically interesting case  $\mathbb{K} = \mathbb{R}$ , although by introducing some insignificant and inessential complications this can be extended to the case of a quite arbitrary field  $\mathbb{K}$ .

Recall (see Lecture 19 in [1]) that from any vector space  $\mathcal{V}$  over the field  $\mathbb{R}$  we can construct a vector space  $\mathcal{V}^{\mathbb{C}}$  over the field  $\mathbb{C}$  called a *complexification* of the space  $\mathcal{V}$ . This space possesses the property that each of its vectors  $z$  can be uniquely represented as

$$z = x + iy,$$

where  $x \in \mathcal{V}$  and  $y \in \mathcal{V}$ . For every linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  therefore the formula

$$A^{\mathbb{C}}(x + iy) = Ax + iAy$$

correctly defines some operator  $A^{\mathbb{C}}: \mathcal{V}^{\mathbb{C}} \rightarrow \mathcal{V}^{\mathbb{C}}$ . Since for any number  $a + ib \in \mathbb{C}$  and any vector  $z = x + iy \in \mathcal{V}^{\mathbb{C}}$

$$(a + ib)(x + iy) = (ax - by) + i(ay + bx),$$

we have

$$\begin{aligned} A^{\mathbb{C}}((a + ib)(\mathbf{x} + i\mathbf{y})) &= A(a\mathbf{x} - b\mathbf{y}) + iA(a\mathbf{y} + b\mathbf{x}) = \\ &= (aA\mathbf{x} - bA\mathbf{y}) + i(aA\mathbf{y} + bA\mathbf{x}) = \\ &= (a + ib)(A\mathbf{x} + iA\mathbf{y}) = \\ &= (a + ib)A^{\mathbb{C}}(\mathbf{x} + i\mathbf{y}), \end{aligned}$$

so that  $A^{\mathbb{C}}(c\mathbf{z}) = cA^{\mathbb{C}}\mathbf{z}$ . It is still easier to verify that

$$A^{\mathbb{C}}(\mathbf{z}_1 + \mathbf{z}_2) = A^{\mathbb{C}}\mathbf{z}_1 + A^{\mathbb{C}}\mathbf{z}_2$$

for any vectors  $\mathbf{z}_1, \mathbf{z}_2 \in \mathcal{V}^{\mathbb{C}}$ . Hence the operator  $A^{\mathbb{C}}$  is linear.  $\square$

**Definition 1.** An operator  $A^{\mathbb{C}}$  is called a *complexification* of an operator  $A$ .

As we know (see Proposition 1 of Lecture 19 in [1]; recall that in terms of the proposition  $(\mathcal{V}^{\mathbb{C}})^{\mathbb{R}} = \mathcal{V}$ ), any basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  is also a basis of a space  $\mathcal{V}^{\mathbb{C}}$ . It follows that in every such (“real”) basis the matrix of an operator  $A^{\mathbb{C}}$  coincides with the matrix of an operator  $A$ . In this sense the *matrix of an operator is not affected when the operator undergoes complexification*. Hence, in particular, the operators  $A$  and  $A^{\mathbb{C}}$  have the same characteristic polynomial:

$$(1) \quad f_A(\lambda) = f_{A^{\mathbb{C}}}(\lambda).$$

It follows immediately, among other things, that the *Hamilton-Cayley theorem* (Theorem 3 of the preceding lecture) is true for any operators  $A: \mathcal{V} \rightarrow \mathcal{V}$ . Indeed, the proof implies that the theorem is true for the operator  $A^{\mathbb{C}}$  and the operator  $f_A(A)$  is obviously a restriction of the operator  $f_A(A^{\mathbb{C}}) = f_{A^{\mathbb{C}}}(A^{\mathbb{C}})$  to  $\mathcal{V} = \text{Re } \mathcal{V}^{\mathbb{C}}$ . Therefore, since  $f_{A^{\mathbb{C}}}(A^{\mathbb{C}}) = 0$ , we have  $f_A(A) = 0$ .  $\square$

In view of (1) the operators  $A$  and  $A^{\mathbb{C}}$  have the same characteristic roots. These are all the eigenvalues of the operator  $A^{\mathbb{C}}$ , but only those of them that are real are eigenvalues of the operator  $A$ .

If the operator  $A$  is nilpotent, then the operator  $A^{\mathbb{C}}$  is also nilpotent (and has the same degree of nilpotency) and therefore all its eigenvalues are equal to zero.

Since these eigenvalues exhaust all the roots of the polynomial  $f_{A^{\mathbb{C}}} = f_A$ , this proves that  $f_A(\lambda) = (-1)^n \lambda^n$  for any nilpotent operator  $A$ . In terms of matrices this means (we replace  $\lambda$  by  $-\lambda$ ) that *for any nilpotent matrix  $A$  we have the identity*

$$\det(A + \lambda E) = \lambda^n. \quad \square$$

It is apparently very difficult to prove this “purely matrix” statement by a straightforward calculation of the determinant.

By virtue of the Hamilton-Cayley theorem it follows that *in an  $n$ -dimensional space the degree of nilpotency of an arbitrary nilpotent operator does not exceed  $n$* .  $\square$

These beautiful statements show what a powerful tool for proving theorems is a quite trivial, one would think, method of complexification.

Now we shall apply it to the investigation of characteristic roots.

It is obvious that for any subspace  $\mathcal{Q}$  of a space  $\mathcal{V}^{\mathbb{C}}$  the subset  $\operatorname{Re} \mathcal{Q}$  of all vectors of  $\mathcal{V}$  of the form  $\operatorname{Re} z$ , where  $z \in \mathcal{Q}$ , or equivalently (since  $\operatorname{Im} z = \operatorname{Re}(-iz)$ ) of the form  $\operatorname{Im} z$ , where  $z \in \mathcal{Q}$ , is a subspace of the space  $\mathcal{V}$  (if  $x_1 = \operatorname{Re} z_1$ ,  $x_2 = \operatorname{Re} z_2$ , then  $x_1 + x_2 = \operatorname{Re}(z_1 + z_2)$  and  $k \operatorname{Re} z = \operatorname{Re} kz$  for any  $k \in \mathbb{R}$ ).

Similarly, for any subspace  $\mathcal{P}$  of the space  $\mathcal{V}$  the set  $\mathcal{P}^{\mathbb{C}}$  of all vectors of the form  $x + iy$ , where  $x, y \in \mathcal{P}$ , is a subspace of the space  $\mathcal{V}^{\mathbb{C}}$  (it is none other but the span of the subspace  $\mathcal{P}$  in the space  $\mathcal{V}^{\mathbb{C}}$ ). It is clear that

$$\operatorname{Re} \mathcal{P}^{\mathbb{C}} = \mathcal{P}$$

for any subspace  $\mathcal{P} \subset \mathcal{V}$ .

Note that any basis  $e_1, \dots, e_p$  of a subspace  $\mathcal{P}$  (over  $\mathbb{R}$ ) is a basis of the subspace  $\mathcal{P}^{\mathbb{C}}$  (over  $\mathbb{C}$ ) as well.

Now let  $A: \mathcal{V} \rightarrow \mathcal{V}$  be an arbitrary linear operator on  $\mathcal{V}$  and  $A^{\mathbb{C}}: \mathcal{V}^{\mathbb{C}} \rightarrow \mathcal{V}^{\mathbb{C}}$  its complexification. Consider an arbitrary

trary characteristic root  $\lambda$  of the operator  $A$ . It is an eigenvalue of the operator  $A^{\mathbb{C}}$  and therefore the corresponding proper subspace  $\mathcal{Q}_{\lambda}$  is defined in  $\mathcal{V}^{\mathbb{C}}$ .

Suppose first that the root  $\lambda$  is real. Then it is an eigenvalue of the operator  $A$  and the corresponding proper subspace  $\mathcal{P}_{\lambda}$  is defined in the space  $\mathcal{V}$ .

If we are given some system of  $n$  homogeneous linear equations in  $n$  unknowns whose coefficients are real and constitute a matrix of rank  $r$ , then the solutions of the system form in  $\mathbb{R}^n$  a subspace  $\mathcal{P}$  of dimension  $n - r$ , so that each solution is a linear combination of some  $n - r$  linearly independent solutions constituting a basis of that subspace. As already said, it is customary to call this basis a *fundamental system of solutions*.

The same system of equations may be regarded as a system with complex coefficients and its solutions sought in  $\mathbb{C}^n = (\mathbb{R}^n)^{\mathbb{C}}$ . Then every fundamental system of solutions remains a fundamental system of solutions but, to obtain all solutions, one has to take linear combinations of solutions to this system not with real but with any complex coefficients. In terms of the notation introduced above this means that  $\mathcal{P}^{\mathbb{C}}$  is the subspace of solutions of a given system of equations in the space  $\mathbb{C}^n$ .

These general considerations apply in particular to the subspace  $\mathcal{P}_{\lambda}$ , the coordinates of whose vectors in an arbitrary basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  satisfy a system of homogeneous linear equations with a real coefficient matrix  $A - \lambda E$ . As we know, the same vectors of  $\mathbf{e}_1, \dots, \mathbf{e}_n$  constitute a basis of the space  $\mathcal{V}^{\mathbb{C}}$  and in that basis the coordinates of the vectors of  $\mathcal{Q}_{\lambda}$  are defined by the same system of equations. This proves that *for every real characteristic root  $\lambda$  of the operator  $A$  we have*

$$\mathcal{Q}_{\lambda} = \mathcal{P}_{\lambda}^{\mathbb{C}}$$

*and therefore also*

$$(2) \quad \mathcal{P}_{\lambda} = \operatorname{Re} \mathcal{Q}_{\lambda}.$$

Now let  $\lambda$  be nonreal. In that case we *define* a subspace  $\mathcal{P}_{\lambda} \subset \mathcal{V}$  by (2).

Thus subspaces  $\mathcal{P}_\lambda$  are now defined for any characteristic roots  $\lambda$  of the operator  $A$ , the notation in the case of real  $\lambda$  having the former meaning.

A subspace  $\mathcal{P}_\lambda = \text{Re } \mathcal{Q}_\lambda$  will be called a *proper subspace* of the operator  $A$  *belonging to a characteristic root*  $\lambda$ . It should be remembered that its vectors are eigenvectors of the operator  $A$  only when  $\lambda$  is real.

It is clear that each of the spaces  $\mathcal{P}_\lambda$  is invariant under  $A$ .

With  $\lambda$  real,  $\mathcal{P}_\lambda^{\mathbb{C}} = \mathcal{Q}_\lambda$ . What is  $\mathcal{P}_\lambda^{\mathbb{C}}$  equal to when  $\lambda$  is nonreal?

To answer the question we remark that since the coefficients of a characteristic polynomial  $f_A$  are real, besides  $\lambda$  its root is the complex conjugate number  $\bar{\lambda}$ . The coordinates of the vectors in the corresponding proper subspace  $\mathcal{Q}_{\bar{\lambda}}$  of the operator  $A^{\mathbb{C}}$  are solutions of a system of linear equations with a complex conjugate matrix  $A - \bar{\lambda}E$  and hence are obtained from the coordinates of vectors of the subspace  $\mathcal{Q}_\lambda$  by changing to complex conjugate numbers. In coordinate-free terms this means that if  $z = x + iy \in \mathcal{Q}_\lambda$ , then  $\bar{z} = x - iy \in \mathcal{Q}_{\bar{\lambda}}$ . We can write this fact as

$$\mathcal{Q}_{\bar{\lambda}} = \overline{\mathcal{Q}_\lambda},$$

a convention but a clear one.

It follows that  $\text{Re } \mathcal{Q}_{\bar{\lambda}} = \text{Re } \mathcal{Q}_\lambda$ , i.e. that

$$\mathcal{P}_\lambda = \mathcal{P}_{\bar{\lambda}}.$$

On the other hand, since  $\lambda \neq \bar{\lambda}$ , the sum of subspaces  $\mathcal{Q}_\lambda$  and  $\mathcal{Q}_{\bar{\lambda}}$  is (Proposition 1 of Lecture 14) their direct sum  $\mathcal{Q}_\lambda \otimes \mathcal{Q}_{\bar{\lambda}}$ . If  $e_1, \dots, e_q$  is a basis of the subspace  $\mathcal{Q}_\lambda$ , where  $q = \dim \mathcal{Q}_\lambda$ , then the vectors of  $e_1, \dots, e_q, \bar{e}_1, \dots, \bar{e}_q$  obviously constitute a basis of the space  $\mathcal{Q}_\lambda \otimes \mathcal{Q}_{\bar{\lambda}}$ . But then so do the vectors

$$\begin{aligned} \text{Re } e_1 &= \frac{e_1 + \bar{e}_1}{2}, & \text{Im } e_1 &= \frac{e_1 - \bar{e}_1}{2i} \\ &\dots\dots\dots & & \\ \text{Re } e_q &= \frac{e_q + \bar{e}_q}{2}, & \text{Im } e_q &= \frac{e_q - \bar{e}_q}{2i}. \end{aligned}$$

The vectors  $\operatorname{Re} \mathbf{e}_1, \operatorname{Im} \mathbf{e}_1, \dots, \operatorname{Re} \mathbf{e}_q, \operatorname{Im} \mathbf{e}_q$  are real by construction, i.e. lie in  $\mathcal{V}$ . If  $\mathcal{P} \subset \mathcal{V}$  is a  $2q$ -dimensional subspace of the space  $\mathcal{V}$ , generated by these vectors, then by definition

$$\mathcal{P}^{\mathbb{C}} = \mathcal{Q}_{\lambda} \oplus \mathcal{Q}_{\bar{\lambda}}.$$

But it is easy to see that under the correspondence  $\mathcal{Q} \mapsto \operatorname{Re} \mathcal{Q}$  a sum of subspaces becomes a sum, i.e.

$$\operatorname{Re} (\mathcal{Q}_1 + \mathcal{Q}_2) = \operatorname{Re} \mathcal{Q}_1 + \operatorname{Re} \mathcal{Q}_2$$

for any subspaces  $\mathcal{Q}_1, \mathcal{Q}_2 \subset \mathcal{V}^{\mathbb{C}}$ . Therefore

$$\mathcal{P} = \operatorname{Re} \mathcal{P}^{\mathbb{C}} = \operatorname{Re} (\mathcal{Q}_{\lambda} \oplus \mathcal{Q}_{\bar{\lambda}}) = \operatorname{Re} \mathcal{Q}_{\lambda} + \operatorname{Re} \mathcal{Q}_{\bar{\lambda}} = \mathcal{P}_{\lambda} + \mathcal{P}_{\bar{\lambda}} = \mathcal{P}_{\lambda}.$$

This proves that *for any nonreal characteristic root  $\lambda$  of the operator  $\mathbf{A}$  the equation*

$$\mathcal{P}_{\lambda}^{\mathbb{C}} = \mathcal{Q}_{\lambda} \oplus \mathcal{Q}_{\bar{\lambda}}.$$

*is valid.*  $\square$

The example of the subspaces  $\mathcal{Q}_1 = \mathcal{Q}_{\lambda}$  and  $\mathcal{Q}_2 = \mathcal{Q}_{\bar{\lambda}}$  shows that under the correspondence  $\mathcal{Q} \mapsto \operatorname{Re} \mathcal{Q}$  a direct sum does not necessarily become a direct sum. It is easy to see, however, that *if  $\mathcal{Q}_1 = \mathcal{P}_1^{\mathbb{C}}$  and  $\mathcal{Q}_2 = \mathcal{P}_2^{\mathbb{C}}$ , then*

$$(3) \quad \operatorname{Re} (\mathcal{Q}_1 \oplus \mathcal{Q}_2) = \operatorname{Re} \mathcal{Q}_1 \oplus \operatorname{Re} \mathcal{Q}_2.$$

Indeed, it is clear that

$$(\mathcal{P}_1 \oplus \mathcal{P}_2)^{\mathbb{C}} = \mathcal{P}_1^{\mathbb{C}} \oplus \mathcal{P}_2^{\mathbb{C}}.$$

Therefore, by applying  $\operatorname{Re}$  we obtain (3).  $\square$

We now prove for spaces  $\mathcal{P}_{\lambda}$  the analogue of Proposition 1 of Lecture 14.

**Proposition 1.** *Let  $\lambda_1, \dots, \lambda_m$  be characteristic roots of an operator  $\mathbf{A}$  (whether real or not) such that*

$$\lambda_i \neq \lambda_j \text{ and } \lambda_i \neq \bar{\lambda}_j, \quad i, j = 1, \dots, m,$$

*with  $i \neq j$ . Then the sum  $\mathcal{P}$  of the subspaces  $\mathcal{P}_{\lambda_1}, \dots, \mathcal{P}_{\lambda_m}$  is their direct sum:*

$$(4) \quad \mathcal{P} = \mathcal{P}_{\lambda_1} \oplus \dots \oplus \mathcal{P}_{\lambda_m}.$$

The example of the subspaces  $\mathcal{P}_\lambda$  and  $\mathcal{P}_{\bar{\lambda}}$  shows that the condition  $\lambda_i \neq \bar{\lambda}_j$  is essential here.

*Proof.* Let  $\lambda_1, \dots, \lambda_r$  be all the given real roots and  $\lambda_{r+1}, \dots, \lambda_m$  all the nonreal ones. Then the  $2m - r$  eigenvalues

$$\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \bar{\lambda}_{r+1}, \dots, \lambda_m, \bar{\lambda}_m$$

of the operator  $A^{\mathbb{C}}$  are all distinct and therefore the sum  $\mathcal{Q}$  of the corresponding proper subspaces is their direct sum:

$$\mathcal{Q} = \mathcal{Q}_{\lambda_1} \oplus \dots \oplus \mathcal{Q}_{\lambda_r} \oplus (\mathcal{Q}_{\lambda_{r+1}} \oplus \mathcal{Q}_{\bar{\lambda}_{r+1}}) \oplus \dots \oplus (\mathcal{Q}_{\lambda_m} \oplus \mathcal{Q}_{\bar{\lambda}_m}).$$

Applying Re and taking into account the fact that

$$\mathcal{Q}_{\lambda_1} = \mathcal{P}_{\lambda_1}^{\mathbb{C}}, \dots, \mathcal{Q}_{\lambda_r} = \mathcal{P}_{\lambda_r}^{\mathbb{C}},$$

$$\mathcal{Q}_{\lambda_{r+1}} \oplus \mathcal{Q}_{\bar{\lambda}_{r+1}} = \mathcal{P}_{\lambda_{r+1}}^{\mathbb{C}}, \dots, \mathcal{Q}_{\lambda_m} \oplus \mathcal{Q}_{\bar{\lambda}_m} = \mathcal{P}_{\lambda_m}^{\mathbb{C}},$$

we obtain immediately equation (4).  $\square$

In the case where the operator  $A^{\mathbb{C}}$  is diagonalizable it follows from Proposition 1 that the space  $\mathcal{V}$  is decomposable as a direct sum of invariant (under  $A$ ) spaces  $\mathcal{P}_\lambda$ , where  $\lambda$  runs over all the real roots of the polynomial  $f_A$  and all of its mutually nonconjugate nonreal roots.

The restriction  $A_\lambda = A|_{\mathcal{P}_\lambda}$  of the operator  $A$  to a subspace  $\mathcal{P}_\lambda$ , with  $\lambda$  real, is known, it is the scalar operator  $\lambda E$  having a diagonalizable (scalar) matrix  $\lambda E$  in an arbitrary basis.

Consider now the operator  $A_\lambda = A|_{\mathcal{P}_\lambda}$ , with  $\lambda$  nonreal. Let

$$\lambda = \alpha + i\beta, \text{ where } \alpha, \beta \in \mathbb{R} \text{ and } \beta \neq 0.$$

It was shown above that for any basis  $e_1, \dots, e_q$  of a subspace  $\mathcal{Q}_\lambda$  the vectors

$$(5) \quad \operatorname{Re} e_1, \operatorname{Im} e_1, \dots, \operatorname{Re} e_q, \operatorname{Im} e_q$$

constitute a basis of a space  $\mathcal{P}_\lambda$ . Set, to simplify notation,  $e = e_1$ ,  $x = \operatorname{Re} e$ ,  $y = \operatorname{Im} e$  and consider a two-dimensional subspace  $\mathcal{P} \subset \mathcal{P}_\lambda$  with a basis  $x, y$ .

Since  $\mathbf{e} \in \mathcal{Q}_\lambda$ , we have  $A^\mathbb{C} \mathbf{e} = \lambda \mathbf{e}$ , i.e.

$$A^\mathbb{C} (\mathbf{x} + i\mathbf{y}) = (\alpha + i\beta) (\mathbf{x} + i\mathbf{y}).$$

This means by definition that

$$A\mathbf{x} + iA\mathbf{y} = (\alpha\mathbf{x} - \beta\mathbf{y}) + i(\beta\mathbf{x} + \alpha\mathbf{y}),$$

i.e. that

$$A\mathbf{x} = \alpha\mathbf{x} - \beta\mathbf{y},$$

$$A\mathbf{y} = \beta\mathbf{x} + \alpha\mathbf{y}.$$

Thus we see that a subspace  $\mathcal{P}$  is invariant under the operator  $A$  and that the restriction of the operator  $A$  to  $\mathcal{P}$  is a matrix

$$(6) \quad \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$$

in the basis  $\mathbf{x}, \mathbf{y}$ .  $\square$

Since a space  $\mathcal{P}_\lambda$  is a direct sum  $q$  of subspaces of the form  $\mathcal{P}$ , we see that in the basis (5) the matrix of the operator  $A_\lambda$  is a block

$$\begin{pmatrix} \begin{array}{cc|ccc} \alpha_1 & \beta_1 & & & \\ -\beta_1 & \alpha_1 & & & \\ \hline & & \alpha_2 & \beta_2 & \\ & & -\beta_2 & \alpha_2 & \\ & & \hline & & & \ddots & \\ & & & & \alpha_q & \beta_q \\ & & & & -\beta_q & \alpha_q \end{array} & & 0 \end{pmatrix}$$

with  $q = \dim \mathcal{Q}_\lambda$  matrices of the form (6) in the diagonal.

Comparing all that which has been proved we see that the following theorem is true.

**Theorem 1.** For a linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  over the field  $\mathbb{R}$ , let the operator  $A^\mathbb{C}: \mathcal{V}^\mathbb{C} \rightarrow \mathcal{V}^\mathbb{C}$  be diagonalizable (this is in particular the case if the operator  $A$  has a simple spectrum). Also let  $\lambda_1, \dots, \lambda_r$  be all the real and  $\lambda_{r+1} = \alpha_1 + i\beta_1, \dots, \lambda_m = \alpha_{m-r} + i\beta_{m-r}$  be all the nonreal characteristic roots of the operator  $A$ , mutually complex non-conjugate, each of which is repeated as many times as is its multiplicity (so that  $m = 2n - r$ ).



Then the space  $\mathcal{V}$  has a basis in which the matrix of the operator  $A$  is a direct sum of a certain number of first order matrices  $\lambda_1, \dots, \lambda_r$  and second order matrices

$$\begin{pmatrix} \alpha_1 & \beta_1 \\ -\beta_1 & \alpha_1 \end{pmatrix}, \dots, \begin{pmatrix} \alpha_{m-r} & \beta_{m-r} \\ -\beta_{m-r} & \alpha_{m-r} \end{pmatrix},$$

i.e. is of the form

(7) 
$$\begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_r & & & & & & \\ & & & \ddots & & & & & \\ & & & & \alpha_1 & \beta_1 & & & \\ & & & & -\beta_1 & \alpha_1 & & & \\ & & & & & \ddots & & & \\ & & & & & & \ddots & & \\ & & & & & & & \alpha_{m-r} & \beta_{m-r} \\ & & & & & & & -\beta_{m-r} & \alpha_{m-r} \\ 0 & & & & & & & & & \end{pmatrix} \cdot \square$$

Of course, a similar theorem, but with a more complicated matrix (7), holds also when the operator  $A^{\mathbb{C}}$  is nondiagonalizable. We shall not need the theorem and therefore we shall neither prove it nor state it.

# Lecture 18

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*Euclidean and unitary spaces. Orthogonal complements. The identification of vectors and covectors. Annulets and orthogonal complements. Bilinear functionals and linear operators. Elimination of arbitrariness in the identification of tensors of different types. The metric tensor. Lowering and raising of indices*

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According to Definitions 2 and 3 of Lecture 13 in [1] a vector space  $\mathcal{V}$  over the field  $\mathbb{R}$  is said to be *Euclidean* if some positive definite symmetric bilinear functional is given in it. The functional is called a *scalar multiplication* and its value on vectors  $\mathbf{x}$  and  $\mathbf{y}$ , the *scalar product* of the vectors, is designated by  $\mathbf{x}\mathbf{y}$  or  $(\mathbf{x}, \mathbf{y})$ .

A direct transfer of these concepts to the case of the ground field  $\mathbb{C}$  is impossible, since there is no notion of positivity in  $\mathbb{C}$ . One has to proceed in a more intricate way.

**Definition 1.** A functional  $\mathbf{x}, \mathbf{y} \mapsto B(\mathbf{x}, \mathbf{y})$  given in a complex vector space  $\mathcal{V}$  is said to be *sesquilinear* if it is linear in the first argument, i.e.

$$B(\mathbf{x}_1 + \mathbf{x}_2, \mathbf{y}) = B(\mathbf{x}_1, \mathbf{y}) + B(\mathbf{x}_2, \mathbf{y})$$

and

$$B(c\mathbf{x}, \mathbf{y}) = cB(\mathbf{x}, \mathbf{y})$$

for any vectors  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}, \mathbf{y} \in \mathcal{V}$  and any number  $c \in \mathbb{C}$ , and *semilinear* in the second argument, i.e.

$$B(\mathbf{x}, \mathbf{y}_1 + \mathbf{y}_2) = B(\mathbf{x}, \mathbf{y}_1) + B(\mathbf{x}, \mathbf{y}_2)$$

and

$$B(\mathbf{x}, c\mathbf{y}) = \bar{c}B(\mathbf{x}, \mathbf{y})$$

for any vectors  $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2, \mathbf{y} \in \mathcal{V}$  and any number  $c \in \mathbb{C}$ .

A sesquilinear functional  $B$  is said to be *Hermitian* if

$$B(y, x) = \overline{B(x, y)}$$

for any vectors  $x, y \in \mathcal{V}$ .

For a Hermitian functional, a number  $B(x, x)$  is real whatever the vector  $x \in \mathcal{V}$ . Therefore the question of its sign is meaningful.

A Hermitian functional  $B$  is said to be *positive definite* if

$$B(x, x) > 0$$

for any nonzero vector  $x$  of a space  $\mathcal{V}$ .

A vector space  $\mathcal{V}$  over the field  $\mathbb{C}$  is said to be *unitary* (as well as *Hermitian*) if some positive definite Hermitian sesquilinear functional is given in it. The functional is called a *scalar multiplication* and its value of vectors  $x$  and  $y$ , the *scalar product* of the vectors, is designated by  $xy$  or  $(x, y)$ .

An example of unitary space is the space  $\mathbb{C}^n$  with the scalar product

$$(x, y) = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n.$$

At its early stages the theory of unitary spaces closely resembles that of Euclidean spaces (see Lectures 13 and 14 in [1]). Thus, for example, in unitary space the *length*  $|x| = \sqrt{(x, x)}$  of any vector  $x$  is defined, the Cauchy-Buniakowski inequality is correct, the concepts of *orthogonal* vectors  $((x, y) = 0)$  and of orthonormal families of vectors and, in particular, of *orthonormal bases* make sense, the Bessel inequality holds (Proposition 2 of Lecture 14 in [1]; but one, naturally, has to write  $|x_i|^2$  instead of  $x_i^2$ ), the analogue of Proposition 3 of Lecture 14 in ([1] on the properties of orthonormal bases holds (but say Parseval's formula is now of the form  $(x, y) = x_1 \bar{y}_1 + \dots + \bar{y}_n$ ), the Gram-Schmidt orthogonalization process is applicable and so on. Of course, identically formulated theorems have as a rule different geometrical meaning for Euclidean and unitary spaces. For example, for Euclidean spaces the fact that there exists an orthonormal basis means that any  $n$ -dimensional Euclidean space is isomorphic to the space  $\mathbb{R}^n$  with the multiplication  $(x, y) = x_1 y_1 + \dots + x_n y_n$ ,

while for unitary spaces it means that any  $n$ -dimensional unitary space is isomorphic to the space  $\mathbb{C}^n$  with the multiplication  $(\mathbf{x}, \mathbf{y}) = x_1 \bar{y}_1 + \dots + x_n \bar{y}_n$ .

In what follows we shall prove theorems for Euclidean and unitary spaces simultaneously whenever possible. In contrast to [1] we shall now prefer to designate a scalar product by the symbol  $(\mathbf{x}, \mathbf{y})$ .

Let  $S$  be an arbitrary subset of a Euclidean or unitary space  $\mathcal{V}$ .

**Definition 2.** The *orthogonal complement*  $S^\perp$  of a subspace  $S$  is the set of all vectors of  $\mathcal{V}$  orthogonal to each vector of  $S$ :

$$S^\perp = \{\mathbf{y} \in \mathcal{V}; (\mathbf{x}, \mathbf{y}) = 0, \mathbf{x} \in S\}.$$

The properties of orthogonal complements are similar to those of annulets (see Lecture 4). It is clear, for example, that the *orthogonal complement of any set is a subspace and that  $S^\perp \supset T^\perp$  if  $S \subset T$* . The analogue of Proposition 3 of Lecture 4 also holds:

**Proposition 1.** *The orthogonal complement of an arbitrary set  $S$  coincides with the orthogonal complement of its linear span:*

$$S^\perp = [S]^\perp.$$

*Proof.* Since  $S \subset [S]$ , we have  $S^\perp \supset [S]^\perp$ . Conversely, if  $\mathbf{y} \in S^\perp$  and  $\mathbf{x} = k_1 \mathbf{x}_1 + \dots + k_m \mathbf{x}_m$ , where  $\mathbf{x}_1, \dots, \mathbf{x}_m \in S$ , then

$$(\mathbf{x}, \mathbf{y}) = k_1 (\mathbf{x}_1, \mathbf{y}) + \dots + k_m (\mathbf{x}_m, \mathbf{y}) = 0$$

and hence  $\mathbf{y} \in [S]^\perp$ .  $\square$

Therefore we may consider without loss of generality only the orthogonal complements of subspaces.

The analogue of Proposition 4 of Lecture 4, on the dimension of an annulet, also holds for orthogonal complements. For these, however, a stronger statement is true, which is possible because for every subspace  $\mathcal{P} \subset \mathcal{V}$  its orthogonal complement is a subspace of the same space  $\mathcal{V}$ .

**Proposition 2.** *For any subspace  $\mathcal{P} \subset \mathcal{V}$ , the space  $\mathcal{V}$  is a direct sum of the subspace  $\mathcal{P}$  and its orthogonal complement:*

$$\mathcal{V} = \mathcal{P} \oplus \mathcal{P}^\perp,$$

*Proof.* Let  $\mathbf{e}_1, \dots, \mathbf{e}_p$  be an orthonormal basis of a subspace  $\mathcal{P}$  and let  $\mathbf{x} \in \mathcal{V}$ . Denote the Fourier coefficients of the vector  $\mathbf{x}$  with respect to the orthonormal family of vectors  $\mathbf{e}_1, \dots, \mathbf{e}_p$  by  $x_1, \dots, x_p$  and compose a vector  $\mathbf{x}' = x_1\mathbf{e}_1 + \dots + x_p\mathbf{e}_p$ . Then, according to Proposition 2 of Lecture 14 in [1], the vector  $\mathbf{x} - \mathbf{x}'$  will be orthogonal to all the vectors of  $\mathbf{e}_1, \dots, \mathbf{e}_p$  and therefore (Proposition 1) lie in the subspace  $\mathcal{P}^\perp$ . Thus

$$\mathbf{x} = \mathbf{x}' + (\mathbf{x} - \mathbf{x}'),$$

where  $\mathbf{x}' \in \mathcal{P}$  and  $\mathbf{x} - \mathbf{x}' \in \mathcal{P}^\perp$ . This proves that  $\mathcal{V}' = \mathcal{P} + \mathcal{P}^\perp$ .

It remains to prove that  $\mathcal{P} \cap \mathcal{P}^\perp = \mathbf{0}$ . But this is obvious, since if  $\mathbf{x} \in \mathcal{P} \cap \mathcal{P}^\perp$  and hence  $(\mathbf{x}, \mathbf{y}) = 0$  for any  $\mathbf{y} \in \mathcal{P}$ , then in particular  $(\mathbf{x}, \mathbf{x}) = 0$  and consequently  $\mathbf{x} = \mathbf{0}$ .  $\square$

It should not be surprising now that the analogue of Proposition 5 of Lecture 4 also holds.

**Proposition 3.** *For any subspace  $\mathcal{P} \subset \mathcal{V}'$  we have*

$$\mathcal{P}^{\perp\perp} = \mathcal{P}.$$

*Proof.* If  $\mathbf{x} \in \mathcal{P}$ , then  $(\mathbf{x}, \mathbf{y}) = 0$  for any  $\mathbf{y} \in \mathcal{P}^\perp$  and therefore  $(\mathbf{x}, \mathbf{y}) = 0$ ; this precisely means that  $\mathbf{x} \in \mathcal{P}^{\perp\perp}$ . Consequently,  $\mathcal{P} \subset \mathcal{P}^{\perp\perp}$ . Conversely, let  $\mathbf{x} \in \mathcal{P}^{\perp\perp}$ . Using Proposition 2 set  $\mathbf{x} = \mathbf{x}' + \mathbf{x}''$ , where  $\mathbf{x}' \in \mathcal{P}$ ,  $\mathbf{x}'' \in \mathcal{P}^\perp$  and therefore  $(\mathbf{x}', \mathbf{x}'') = 0$ . Since  $\mathbf{x} \in \mathcal{P}^{\perp\perp}$ , we have  $(\mathbf{x}, \mathbf{x}'') = 0$  and hence

$$(\mathbf{x}'', \mathbf{x}'') = (\mathbf{x} - \mathbf{x}', \mathbf{x}'') = (\mathbf{x}, \mathbf{x}'') - (\mathbf{x}', \mathbf{x}'') = 0.$$

Consequently,  $\mathbf{x}'' = \mathbf{0}$  and therefore  $\mathbf{x} = \mathbf{x}' \in \mathcal{P}$ . Thus  $\mathcal{P} = \mathcal{P}^{\perp\perp}$ .  $\square$

We stress that all this holds for both Euclidean and unitary spaces.

This parallelism between Euclidean and unitary spaces is violated for conjugate spaces. Therefore we have to consider the conjugate space  $\mathcal{V}''$  separately for a Euclidean and a unitary space  $\mathcal{V}$ .

First let  $\mathcal{V}$  be a Euclidean space.

**Proposition 4.** *For any Euclidean space  $\mathcal{V}$ , there is a natural isomorphism*

$$\mathcal{V}' \approx \mathcal{V}.$$

*Proof.* According to Proposition 2 of Lecture 4 it suffices to prove that the space  $\mathcal{V}$  is self-dual:

$$\mathcal{V} \mid \mathcal{V},$$

i.e. that there exists a natural pairing of  $\mathcal{V}$  and  $\mathcal{V}$ . But such a pairing does in fact exist, it is a scalar multiplication (obviously nonsingular, by virtue of positivity).  $\square$

The isomorphism  $\mathcal{V}' \rightarrow \mathcal{V}$  is explicitly defined as a mapping associating with every vector  $\mathbf{y} \in \mathcal{V}$  a linear functional

$$\xi_{\mathbf{y}}: \mathbf{x} \mapsto (\mathbf{x}, \mathbf{y}).$$

It is clear that the correspondence  $\mathbf{y} \mapsto \xi_{\mathbf{y}}$  is a homomorphism. Since the spaces  $\mathcal{V}$  and  $\mathcal{V}'$  have the same dimension, to prove that that homomorphism is an isomorphism, it is sufficient to establish that its kernel is zero, i.e. that if  $\mathbf{y} \neq \mathbf{0}$ , then  $\xi_{\mathbf{y}} \neq 0$ . But this is obvious, since say  $\xi_{\mathbf{y}}(\mathbf{y}) = (\mathbf{y}, \mathbf{y}) \neq 0$ .

Here we have in fact repeated the proof of Proposition 2 of Lecture 5 for the case of the pairing  $\mathbf{x}, \mathbf{y} \mapsto (\mathbf{x}, \mathbf{y})$ .

On the face of it it seems that the proof remains valid for the case of unitary spaces too. But a closer look shows that *for a unitary space  $\mathcal{V}$  the mapping  $\mathbf{y} \mapsto \xi_{\mathbf{y}}$  is not a homomorphism*. That is, although it does obviously carry a sum over into a sum,

$$\xi_{\mathbf{y}_1 + \mathbf{y}_2} = \xi_{\mathbf{y}_1} + \xi_{\mathbf{y}_2}, \quad \mathbf{y}_1, \mathbf{y}_2 \in \mathcal{V},$$

it carries a product by a number over into a product by a complex conjugate number, i.e. for any vector  $\mathbf{y} \in \mathcal{V}$  and any number  $c \in \mathbb{C}$  we have

$$\xi_{c\mathbf{y}} = \bar{c}\xi_{\mathbf{y}}.$$

Mappings of vector spaces over the field  $\mathbb{C}$  possessing these properties are called *semilinear*. It is easy to see that just as a linear mapping, a semilinear mapping of vector spaces of the same dimension, having a zero kernel, is bijective

(is said to be a *semilinear isomorphism*). All the arguments in the proof of Proposition 4 thus remain fully valid and hence *the space  $\mathcal{V}'$  conjugate to a unitary space  $\mathcal{V}$  is semilinearly isomorphic to it.*

This of course suits us little, since the primary importance of Proposition 4 is that it allows identification of every Euclidean space  $\mathcal{V}$  with its conjugate space  $\mathcal{V}'$  (without, in particular, distinguishing—even in notation!—between the vector  $\mathbf{y}$  and the covector  $\xi_{\mathbf{y}}$ ) while the presence of only a semilinear isomorphism in the unitary case permits such identification only with reservations.

This can be remedied by understanding the covectors of  $\mathcal{V}'$  to be not linear but *semilinear functionals*  $\xi: \mathcal{V} \rightarrow \mathbb{C}$ , i.e. mappings such that

$$\xi(\mathbf{x} + \mathbf{y}) = \xi(\mathbf{x}) + \xi(\mathbf{y})$$

and

$$\xi(c\mathbf{x}) = \bar{c}(\xi\mathbf{x})$$

for any vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  and any number  $c \in \mathbb{C}$ . This seems to be the trend, but at present this substitution is not at all generally accepted.

Alternatively, we may define a new operation of multiplication  $\xi \mapsto c\xi$  of functionals  $\xi$  by numbers  $c \in \mathbb{C}$  in the space  $\mathcal{V}'$ , the linear functionals  $\xi: \mathcal{V} \mapsto \mathbb{C}$  assumed as before to be its vectors, putting

$$(c\xi)(\mathbf{x}) = \xi(\bar{c}\mathbf{x}) = \bar{c}(\xi(\mathbf{x}))$$

for any functional  $\xi \in \mathcal{V}'$ , any number  $c \in \mathbb{C}$ , and any vector  $\mathbf{x} \in \mathcal{V}$ .

Of course, thus we simply transfer the “parasite” complex conjugation to other, possibly less conspicuous, places. Since both variants have their advantages and drawbacks and neither has yet become prevalent, each requiring a revision of all the previous material (say, of tensor theory), we shall give up both, stick to the former point of view, and shall not aim at formal perfection.

As far as the identification of vectors and covectors is concerned, we shall allow it to be made in the unitary case

too, remembering all the time the possibility of the boring complex conjugation appearing.

The fact that in Euclidean and unitary spaces vectors and covectors practically coincide allows identification of objects fundamentally different in arbitrary vector spaces.

For example, it is easy to see that when covectors are identified with vectors the annulet  $S^\circ$  of an arbitrary set  $S \subset \mathcal{V}$  coincides with the orthogonal complement  $S^\perp$ . Indeed, the inclusion  $\xi \in S^\circ$  implies that  $\xi(\mathbf{x}) = 0$  for any  $\mathbf{x} \in S$ . Therefore, if we identify a covector  $\xi$  and a vector  $\mathbf{y} \in \mathcal{V}$  satisfying the relation  $\xi(\mathbf{x}) = (\mathbf{x}, \mathbf{y})$  for any vector  $\mathbf{x} \in \mathcal{V}$ , then, in particular, for any vector  $\mathbf{x} \in S$  we shall have the equation  $(\mathbf{x}, \mathbf{y}) = 0$  implying that  $\mathbf{x} \in S^\perp$ .  $\square$

This explains the above parallelism between annulets and orthogonal complements.

The coincidence of vectors and covectors in Euclidean space leads to the most pronounced simplifications in tensor theory, allowing identification of different  $(p, q)$ -tensors with the same sum  $p + q$ , since it is possible to declare each argument of a tensor to be a vector or covector at will.

Consider, for example, a  $(2, 0)$ -tensor, i.e. a bilinear functional  $A: \mathbf{x}, \mathbf{y} \mapsto A(\mathbf{x}, \mathbf{y})$ . Assuming its second argument  $\mathbf{y}$  to be a covector, we obtain from it a bilinear  $(1, 1)$ -functional, i.e. a linear operator  $A: \mathbf{x} \mapsto A\mathbf{x}$ . It is easy to entangle oneself in identifications here. So be attentive: an operator  $A$  transforms a vector  $\mathbf{x}$  into a vector  $A\mathbf{x}$  such that, if considered as a functional on covectors, it has on a covector  $\xi$  the value  $\xi(A\mathbf{x}) = A(\mathbf{x}, \mathbf{y})$ , where  $\mathbf{y}$  is the vector identified with the covector  $\xi$ . But the identification  $\xi = \mathbf{y}$  implies that  $\xi(\mathbf{z}) = (\mathbf{z}, \mathbf{y})$  for any vector  $\mathbf{z} \in \mathcal{V}$  and in particular that  $\xi(A\mathbf{x}) = (A\mathbf{x}, \mathbf{y})$ . Thus

$$(1) \quad A(\mathbf{x}, \mathbf{y}) = (A\mathbf{x}, \mathbf{y}).$$

Formula (1) explicitly describes the bijective correspondence between linear operators  $A: \mathbf{x} \mapsto A\mathbf{x}$  and bilinear functionals  $A: \mathbf{x}, \mathbf{y} \mapsto A(\mathbf{x}, \mathbf{y})$  in a Euclidean space  $\mathcal{V}$ . Irrespective of the general theory it could be accepted as a definition of that correspondence. Then it is necessary to establish that for any linear operator  $A$  the functional  $A$



defined by (1) is bilinear (this reduces to an automatic check), that the resulting “operator”  $\Rightarrow$  “functional” correspondence is a homomorphism of the corresponding vector spaces (another automatic check), that that homomorphism is injective (put  $y = Ax$  and take advantage of the nonsingularity of scalar multiplication) and, finally, that that homomorphism is an isomorphism (it follows from its being an injection, for both vector spaces have the same dimension  $n^2$ ).

The last approach is suitable for unitary spaces as well, but sesquilinear functionals will obviously result instead of bilinear functionals. In order to avoid making such reservations we confine ourselves (in this lecture) to Euclidean spaces; the reader can no doubt make all changes involved in switching to unitary spaces on his own.

An attentive reader must have already noticed that there is an element of arbitrariness in the identification of bilinear functionals<sup>1</sup> and linear operators described above. Indeed, we take the second argument of a bilinear functional  $A(x, y)$  as a covector, but we could be equally well justified (in Euclidean space) in assuming the first argument to be a covector. Then, generally speaking, a different linear operator  $A^*$  would result for which there would hold the formula

$$(2) \quad A(x, y) = (x, A^*y).$$

The situation is similar, and still worse, for tensors of other types. Consider, for example, a  $(3, 1)$ -tensor  $T(x_1, x_2, x_3; \xi_1)$ . By declaring the vector  $x_3$  to be a covector (and denoting it by, say,  $\xi_2$ ) we identify that tensor with a  $(2, 2)$ -tensor  $T(x_1, x_2; \xi_2, \xi_1)$ . But we may assume the new covector argument  $\xi_2$  to be not the first but the second argument, and then, in general, a different  $(2, 2)$ -tensor will result. Moreover, assuming the vector  $x_2$ , rather than  $x_3$ , to be a covector, we may obtain another  $(2, 2)$ -tensor distinct from the first two. We may, for example, declare the argument  $x_1$  to be a covector and simultaneously consider the argument  $\xi_1$  to be a vector! Then a  $(3, 1)$ -tensor results, of the same type as the original one but distinct from it, and so on and so forth.

For definiteness we should introduce a single enumeration (or at least a single ordering) of vector and covector arguments and write them alternately in that order. Thus, for example, the symbol  $T(x_1, x_2, \xi_3, x_4)$  for a  $(3, 1)$ -tensor means that when the covector  $\xi_3$  is declared to be a vector, a  $(4, 0)$ -tensor results in which the new vector argument is the third, and when the vector  $x_2$  ( $x_4$ ) is declared to be a covector, a  $(2, 2)$ -tensor results in which the new covector argument is the first (the second), among the covector arguments.

In order to avoid misunderstanding we stress that the symbols  $T(x_1, x_2, x_3, \xi_4)$  and  $T(x_1, x_2, \xi_3, x_4)$  designate both a  $(3, 1)$ -tensor with three vector and one covector arguments. These tensors differ only in their origin, the first of them having been obtained from some  $(4, 0)$ -tensor  $T(x_1, x_2, x_3, x_4)$  by giving the name of covector to the argument  $x_4$  and the second by declaring the argument  $x_3$  to be a covector. Distinguishing between tensors of the form  $T(x_1, x_2, \xi_3, x_4)$  and those of the form  $T(x_1, x_2, x_3, \xi_4)$  makes no sense in arbitrary vector spaces.

Let  $e_1, \dots, e_n$  be an arbitrary basis of a Euclidean space  $\mathcal{V}$ . Then, by virtue of the identification  $\mathcal{V} = \mathcal{V}'$ , the conjugate basis  $e^1, \dots, e^n$  is also a basis of the space  $\mathcal{V}$ , but, in general, one distinct from the basis  $e_1, \dots, e_n$ . It is connected with the basis  $e_1, \dots, e_n$  by the relations

$$(e_i, e^j) = \delta_i^j, \quad i, j = 1, \dots, n.$$

If

$$e_i = g_{ij} e^j, \quad i, j = 1, \dots, n,$$

are the formulas for the change from the basis  $e^1, \dots, e^n$  to the basis  $e_1, \dots, e_n$ , then

$$(e_i, e_j) = g_{ik} (e^k, e_j) = g_{ik} \delta_j^k = g_{ij},$$

and we see that the numbers  $g_{ij}$  are the familiar *metric coefficients* of the basis  $e_1, \dots, e_n$  (see Lecture 14 in [1]). They constitute a nonsingular matrix whose inverse is a matrix with the elements

$$g^{ij} = (e^i, e^j).$$

If we change to a basis

$$\mathbf{e}_{i'} = c_{i'}^i \mathbf{e}_i,$$

then the metric coefficients  $g_{i'j'}$  of the new basis are expressed by the formulas

$$g_{i'j'} = c_{i'}^i c_{j'}^j g_{ij},$$

i.e. are transformed by tensor law. This means that the numbers  $g_{ij}$  are the coefficients of some tensor

$$G = g_{ij} \mathbf{e}^i \otimes \mathbf{e}^j = g^{ij} \mathbf{e}_i \otimes \mathbf{e}_j$$

called a *metric tensor* of a Euclidean space  $\mathcal{V}$ . The value  $G(\mathbf{x}, \mathbf{y})$  of the tensor on vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  is just the scalar product of the vectors:

$$G(\mathbf{x}, \mathbf{y}) = g_{ij} x^i y^j = (\mathbf{x}, \mathbf{y}).$$

Thus the term metric tensor has exactly the same meaning as the term scalar multiplication!

Now let  $\mathbf{x}$  be an arbitrary vector of the space  $\mathcal{V}$ . By definition its tensor product  $\mathbf{x} \otimes G$  with a metric tensor  $G$  is a (2, 1)-tensor. This can be contracted (see Lecture 6) over the only superscript and over one, say for definiteness the second, subscript (although this is of no importance in the given case). As a result we obtain some (1, 0)-tensor, i.e. a covector  $\xi$ . The value  $\xi(\mathbf{y})$  of the covector on an arbitrary vector  $\mathbf{y}$  is equal to the contraction  $\text{tr}(\xi \otimes \mathbf{y})$  of a tensor product  $\xi \otimes \mathbf{y}$  and hence to the result of the complete contraction  $\mathbf{x} \otimes G \otimes \mathbf{y}$ , i.e. (see the examples of contraction in Lecture 6) to the value  $G(\mathbf{x}, \mathbf{y}) = (\mathbf{x}, \mathbf{y})$  of the tensor  $G$  on the vectors  $\mathbf{x}$  and  $\mathbf{y}$ . Since the equation  $\xi(\mathbf{y}) = (\mathbf{x}, \mathbf{y})$  means by definition that the covector  $\xi$  is identified with the vector  $\mathbf{x}$ , this proves that *the vector  $\mathbf{x}$ , regarded as a covector, is a contraction of the tensor  $\mathbf{x} \otimes G$  or in common parlance is a contraction of the vector  $\mathbf{x}$  with the tensor  $G$ .*  $\square$

In a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  a tensor  $\mathbf{x} \otimes G$  has the coordinates  $g_{ij} x^h$  and its contraction the coordinates

$$x_i = g_{ij} x^j.$$

The numbers  $x_1, \dots, x_n$  are called the *covariant coordinates* of the vector  $\mathbf{x}$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$ . By definition they are the coordinates of the corresponding covector  $\xi$  in the conjugate basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  or, equivalently, the coordinates of the vector  $\mathbf{x}$  in the basis  $\mathbf{e}^1, \dots, \mathbf{e}^n$  whose elements are identified with the vectors of the space  $\mathcal{V}$ . The “actual” coordinates  $x^1, \dots, x^n$  of the vector  $\mathbf{x}$  in the basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are called the *contravariant coordinates* of the vector, to distinguish them from the covariant coordinates.

A change from the coordinates  $x^i$  to the coordinates  $x_i$  is sometimes called the *lowering* of the index  $i$  and the inverse operation is called the *raising* of the index.

According to a single ordering of arguments of an arbitrary tensor (see above), the superscripts and subscripts of its coordinates (components) must also be ordered. Therefore, if there are superscripts and subscripts it is necessary to leave gaps above for places occupied by subscripts and conversely gaps below for places occupied by superscripts. Dots are sometimes put in the gaps for clearness.

Thus, for example, the coordinates of a tensor  $T$  ( $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \xi_1$ ) are designated

$$T_{i_1 i_2 i_3}^{j_1} = T_{i_1 i_2 i_3}^{j_1}$$

and the coordinates of a tensor  $T$  ( $\mathbf{x}_1, \mathbf{x}_2, \xi_1, \mathbf{x}_3$ ) as

$$T_{i_1 i_2}^{j_1}{}_{i_3} = T_{i_1 i_2 i_3}^{j_1}$$

In particular, there are two symbols for the coordinates of a linear operator:  $a_i^j$  and  $a_j^i$ , the first when the operator is obtained from a bilinear functional with the coordinates  $a_{ij}$  by declaring the second argument to be a covector (formula (1)) and the second by declaring the first argument to be a covector (formula (2)). Since

$$a_{ij} = A(\mathbf{e}_i, \mathbf{e}_j), \quad a_i^j = (A\mathbf{e}_i, \mathbf{e}^j), \quad a_j^i = (\mathbf{e}_i, A^*\mathbf{e}^j),$$

we have

$$(3) \quad a_{ij} = g_{ik} a_j^k = g_{kj} a_i^k,$$

and also

$$(4) \quad a_i^j = g^{kj} a_{ik}, \quad a_j^i = g^{ik} a_{kj}.$$

The numbers  $a_{ij}$ ,  $a_i^k$ ,  $a_j^i$  (as well as the numbers  $a^{ij} = g^{ik}g^{jl}a_{kl}$ ) may be regarded as different coordinates of the same mathematical object that, just as a particle in quantum mechanics, has two faces, a “functional” and an “operator” one. The coordinates  $a_{ij}$  are called the coordinates covariant over both indices, the coordinates  $a_j^i$  are called the coordinates covariant over the first index and contravariant over the second and so on.

As shown by (3) and (4), all these coordinates are obtained from one another by tensor multiplication by “reciprocal” tensors  $g_{ij}$  and  $g^{ij}$  followed by contractions over the corresponding indices.

The lowering and raising of indices can be effected in a similar manner for other tensors as well. For example,

$$T_{i_1 i_2}^{j_1 j_2} = g_{i_1 k_1} g^{i_1 l_1} T_{i_2 l_1}^{k_1 j_2}.$$

If a basis  $e_1, \dots, e_n$  is orthonormal, then the conjugate basis  $e^1, \dots, e^n$  coincides with it and all formulas for the lowering and raising of indices simply turn into the equations of the corresponding coordinates (having the same indices regardless of their position). For example,

$$(5) \quad a_{ij} = a_i^j = a_i^j$$

for bilinear functionals and

$$x_i = x^i$$

for vectors. That is why even in the first semester’s lectures we used symbols with subscripts for the coordinates of vectors in an orthonormal basis.

Note that according to the first of the formulas (5) a bilinear functional  $A$  and the linear operator  $A$  corresponding to it according to (1) have the same matrix in every orthonormal basis.

As to the operator  $A^*$  defined by (2), its matrix (in an orthonormal basis) is the transpose of the operator  $A$ .

In what follows we shall always identify bilinear functionals and linear operators by (1), so we shall not need the explicit notation  $a_j^i$  for the elements of the matrix of the linear operator. Therefore we shall continue to designate these elements as  $a_j^i$ .

# Lecture 19

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*Adjoint operators. Self-adjoint operators. Skew-symmetric and skew-Hermitian operators. Analogy between Hermitian operators and real numbers. Spectral properties of self-adjoint operators. The orthogonal diagonalizability of self-adjoint operators*

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According to formulas (1) and (2) of the preceding lecture we may associate with every linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  acting in a Euclidean or unitary space  $\mathcal{V}$  a bilinear functional  $A$  and associate with the latter a linear operator  $A^*$ .

**Definition 1.** The operator  $A^*$  is called the operator *adjoint* to the operator  $A$ . It is uniquely characterized by the relation

$$(1) \quad (Ax, y) = (x, A^*y)$$

which must hold for any vectors  $x, y \in \mathcal{V}$ .

This definition has meaning for a unitary space  $\mathcal{V}$  as well, but while for a Euclidean space  $\mathcal{V}$  the operator  $A^*$  is none other but an adjoint operator  $A': \mathcal{V}' \rightarrow \mathcal{V}'$  regarded, by virtue of the identification  $\mathcal{V} = \mathcal{V}'$ , as an operator on  $\mathcal{V}$ , for a unitary space  $\mathcal{V}$  the operator  $A^*$  differs from the operator  $A'$ , even after the identification of vectors and covectors, in that it is complex conjugate.

In an arbitrary basis  $e_1, \dots, e_n$  of a Euclidean space  $\mathcal{V}$  the elements  $a_j^{*i}$  of the matrix of the operator  $A^*$  are related to the elements  $a_j^i$  of the matrix of the operator  $A$  by the formula

$$a_j^{*i} = g^{ik} g_{jl} a_k^l.$$

For an orthonormal basis  $e_1, \dots, e_n$  this formula takes the form

$$a_j^{*i} = a_i^j.$$

In a unitary space  $\mathcal{V}$  the corresponding formula (in an orthonormal basis) is of the form

$$a_j^{*i} = \overline{a_i^j}.$$

Thus an operator  $A^*$  on a Euclidean (unitary) space  $\mathcal{V}$  is adjoint to an operator  $A$  if and only if in some (and hence in any) orthonormal basis its matrix is the transposed (respectively transposed and complex conjugate) matrix of the operator  $A$ .  $\square$

For the Euclidean case this statement can be proved without any calculations, if one recalls that operators  $A$  and  $A'$  have transposed matrices in conjugate bases (see Lecture 13) and that a basis  $e_1, \dots, e_n$  is orthonormal if and only if it coincides with the conjugate basis  $e^1, \dots, e^n$  regarded as a basis in  $\mathcal{V}$ .

The properties of the adjoint operator  $A^*$  are naturally quite similar to those of the adjoint operator  $A'$ . For example,  $A^{**} = A$  and  $(AB)^* = B^*A^*$ . The only essential difference arises as always in unitary spaces in connection with multiplication by numbers. Namely, if  $(cA)^* = cA^*$  in a Euclidean space, then there again arises a “parasite” complex conjugation in the unitary case:  $(cA)^* = \overline{c}A^*$ .

The following definition essentially uses the fact that operators  $A$  and  $A^*$  act in the same space (and hence does not apply to an operator  $A'$ ).

**Definition 2.** An operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  on a Euclidean or unitary space is said to be *self-adjoint* if  $A^* = A$ , i.e. if for any vectors  $x, y \in \mathcal{V}$  we have

$$(Ax, y) = (x, Ay).$$

Self-adjoint operators are also called *symmetric* (or *symmetrical*) operators in the Euclidean case and *Hermitian* operators in the unitary case.

It is clear that an operator  $A$  on a Euclidean (unitary) space is *symmetric* (*Hermitian*) if and only if the corresponding bilinear (sesquilinear) functional  $A$  is *symmetric* (*Hermitian*).

For example, in a unitary space

$$A(y, x) = (Ay, x) = \overline{(x, Ay)} = \overline{(Ax, y)} = \overline{A(x, y)}. \quad \square$$

A sum of self-adjoint operators and a product of a self-adjoint operator by a real number are obviously self-adjoint operators. This means that *self-adjoint operators form a vector space over the field  $\mathbb{R}$*  (that is a subspace of the space  $\text{Op}(\mathcal{V})$  in the case of a Euclidean space  $\mathcal{V}$ ).

Note that a product of two self-adjoint operators may or may not be a self-adjoint operator. More precisely, *a product  $AB$  of two self-adjoint operators  $A$  and  $B$  is a self-adjoint operator if and only if the operators commute, i.e.  $AB = BA$ .*

Indeed, if  $AB = BA$ , then  $(AB)^* = (BA)^* = A^*B^* = AB$ . Conversely, if  $(AB)^* = AB$ , then  $BA = B^*A^* = (AB)^* = AB$ .  $\square$

A quadratic matrix  $A = (a_{ij}^j)$  consisting of complex numbers is said to be *Hermitian* if after transposing it coincides with the complex conjugate matrix, i.e. if

$$a_{ij}^j = \overline{a_{ji}^i} \text{ for any } i, j = 1, \dots, n.$$

It is clear that *in a Euclidean (unitary) space an operator  $A$  is symmetric (Hermitian) if and only if in some (and hence also in any) orthonormal basis its matrix is symmetric (Hermitian).*  $\square$

**Definition 3.** An operator  $A$  on a Euclidean space  $\mathcal{V}$  is said to be *skew-symmetric* if  $A^* = -A$ , i.e. if

$$(Ax, y) + (x, Ay) = 0$$

for any vectors  $x, y \in \mathcal{V}$ .

Similarly, an operator  $A$  on a unitary space  $\mathcal{V}$  is said to be *skew-Hermitian* if  $A^* = -A$ .

Skew-symmetric operators constitute a quite independent class of linear operators. Skew-symmetric bilinear functionals correspond to them and in coordinates they are characterized by the fact that their matrices are skew-symmetric in every orthonormal basis. They form a subspace in the space  $\text{Op}(\mathcal{V})$ , the space  $\text{Op}(\mathcal{V})$  of all linear operators (cf. Proposition 1 of Lecture 11) being decomposable as a direct sum of the subspaces of symmetric and skew-symmetric



operators, i.e. any linear operator  $A$  can be represented by the sum

$$(2) \quad A = A_{\text{symm}} + A_{\text{skew}}$$

of a symmetric operator  $A_{\text{symm}}$  and a skew-symmetric operator  $A_{\text{skew}}$ , where

$$A_{\text{symm}} = \frac{A + A^*}{2}, \quad A_{\text{skew}} = \frac{A - A^*}{2}.$$

For Hermitian operators the situation is quite different, since skew-Hermitian operators can be reduced in a trivial way to Hermitian operators, a fact having no analogues in Euclidean space. Namely, it follows immediately from the relation  $(iA)^* = \overline{i}A^* = -iA^*$  that *an operator is skew-Hermitian if and only if it has the form  $iA$ , where  $A$  is a Hermitian operator.*  $\square$

At the same time the analogue of decomposition (2) obviously remains valid for operators on unitary space. Therefore *any operator  $A$  on a unitary space  $\mathcal{V}$  can be uniquely represented as*

$$A = B + iC,$$

*where  $B$  and  $C$  are Hermitian operators.* This means (see Definition 1 of Lecture 19 in [1]) *that for any unitary space  $\mathcal{V}$  the vector space  $\text{Op}(\mathcal{V})$  carries the natural structure of a real-complex vector space, the corresponding real subspace being the space of Hermitian operators.*  $\square$

We thus see that in a certain respect Hermitian operators are similar to real numbers. This similarity can be traced in other respects too.

According to Definition 1 of Lecture 11 and the relation, established above for Euclidean spaces, between symmetric bilinear functionals and symmetric linear operators, *in Euclidean space every quadratic functional can be uniquely represented as  $(Ax, x)$ , where  $A$  is some symmetric linear operator.* Functionals of this form present nothing new for non-symmetric linear operators, since  $(Ax, x) = 0$  for all  $x \in \mathcal{V}$ , if (and only if) the operator  $A$  is skew-symmetric.

For unitary spaces the situation turns out to be fundamentally different. This is not surprising, however, for in a unitary space no functional of the form  $(Ax, x)$ , with

$A \neq 0$ , is a quadratic functional in the sense of Definition 1 of Lecture 11 and therefore there are no reasons for the properties of such functionals to resemble those of quadratic functionals.

In a Euclidean space a functional  $(Ax, x)$  could be identically zero without the operator  $A$  being zero. In a unitary space this is not possible.

**Proposition 1.** *If a linear operator  $A: \mathcal{V} \rightarrow \mathcal{V}$  on a unitary space  $\mathcal{V}$  possesses the property that*

$$(3) \quad (Ax, x) = 0$$

*for any vector  $x \in \mathcal{V}$ , then  $A = 0$ .*

*Proof.* Since for any vectors  $x, y \in \mathcal{V}$  we have

$$\begin{aligned} (A(x+y), x+y) &= (Ax, x) + (Ax, y) + (Ay, x) + (Ay, y), \\ (A(x+iy), x+iy) &= \\ &= (Ax, x) + (Ax, iy) + (iAy, x) + (iAy, iy) \end{aligned}$$

and

$$(Ax, iy) = -i(Ax, y), \quad (iAy, x) = i(Ax, y),$$

in view of (3)

$$(Ax, y) + (Ay, x) = 0,$$

$$(Ax, y) - (Ay, x) = 0,$$

and hence  $(Ax, y) = 0$ . Putting here  $y = Ax$ , we have  $(Ax, Ax) = 0$ . Therefore  $Ax = 0$  for any  $x \in \mathcal{V}$ .  $\square$

**Proposition 2 (Hermitian property criterion).** *A linear operator  $A$  on a unitary space  $\mathcal{V}$  is Hermitian if and only if for any vector  $x \in \mathcal{V}$  the number  $(Ax, x)$  is real.*

*Proof.* If the operator  $A$  is Hermitian, then for any vector  $x \in \mathcal{V}$

$$(Ax, x) = (x, Ax) = \overline{(Ax, x)}$$

and hence the number  $(Ax, x)$  is real. Conversely, if  $(Ax, x) = \overline{(Ax, x)}$ , then

$$\begin{aligned} ((A - A^*)x, x) &= (Ax, x) - (A^*x, x) = \\ &= (Ax, x) - (x, Ax) = \\ &= (Ax, x) - \overline{(Ax, x)} = 0, \end{aligned}$$

and, therefore, according to Proposition 1,  $A - A^* = 0$ .  $\square$

The following propositions are true for both Euclidean and unitary spaces (although, in general, each requiring a different proof).

**Proposition 3 (reality).** *All characteristic roots of an arbitrary self-adjoint operator are real.*

*Proof.* Let  $A$  be a self-adjoint operator in a Euclidean or unitary space  $\mathcal{V}$  and let  $\lambda$  be its arbitrary characteristic root.

If the space  $\mathcal{V}$  is unitary (and hence the operator  $A$  is Hermitian), then the number  $\lambda$  is an eigenvalue of the operator  $A$ , i.e. there exists a vector  $\mathbf{x}_0 \neq 0$  such that  $A\mathbf{x}_0 = \lambda\mathbf{x}_0$ . For that vector  $(A\mathbf{x}_0, \mathbf{x}_0) = (\lambda\mathbf{x}_0, \mathbf{x}_0) = \lambda(\mathbf{x}_0, \mathbf{x}_0)$  and hence

$$\lambda = \frac{(A\mathbf{x}_0, \mathbf{x}_0)}{(\mathbf{x}_0, \mathbf{x}_0)}.$$

To complete the proof of Proposition 3 in this case, it remains to note that according to Proposition 2 the right-hand side of this formula is real. Therefore, so is the number  $\lambda$ .

Now let  $\mathcal{V}$  be a Euclidean space. Arguing by contradiction, assume that  $\lambda = \alpha + i\beta$ , where  $\beta \neq 0$ . Then, as was shown in Lecture 16, for an operator  $A$  there exists a two-dimensional invariant subspace  $\mathcal{P}$  in the space  $\mathcal{V}$  and there is a basis  $\mathbf{x}, \mathbf{y}$  in  $\mathcal{P}$  such that

$$\begin{aligned} A\mathbf{x} &= \alpha\mathbf{x} - \beta\mathbf{y}, \\ A\mathbf{y} &= \beta\mathbf{x} + \alpha\mathbf{y}. \end{aligned}$$

Therefore

$$(A\mathbf{x}, \mathbf{y}) = (\alpha\mathbf{x} - \beta\mathbf{y}, \mathbf{y}) = \alpha(\mathbf{x}, \mathbf{y}) - \beta(\mathbf{y}, \mathbf{y})$$

and

$$(\mathbf{x}, A\mathbf{y}) = (\mathbf{x}, \beta\mathbf{x} + \alpha\mathbf{y}) = \beta(\mathbf{x}, \mathbf{x}) + \alpha(\mathbf{x}, \mathbf{y}).$$

Since the operator  $A$  is self-adjoint (symmetric) and hence  $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y})$  it follows that

$$\beta[(\mathbf{x}, \mathbf{x}) + (\mathbf{y}, \mathbf{y})] = 0.$$

Since this last equation is impossible (for  $(\mathbf{x}, \mathbf{x}) > 0$ ,  $(\mathbf{y}, \mathbf{y}) > 0$  and by hypothesis  $\beta \neq 0$ ) this proves that  $\lambda \in \mathbb{R}$ .  $\square$

**Proposition 4 (orthogonality).** *Any two eigenvectors  $\mathbf{x}$  and  $\mathbf{y}$  of a self-adjoint operator  $A$  belonging to different eigenvalues  $\lambda$  and  $\mu$  are orthogonal.*

*Proof.* We have

$$(A\mathbf{x}, \mathbf{y}) = (\lambda\mathbf{x}, \mathbf{y}) = \lambda (\mathbf{x}, \mathbf{y}),$$

$$(\mathbf{x}, A\mathbf{y}) = (\mathbf{x}, \mu\mathbf{y}) = \mu (\mathbf{x}, \mathbf{y})$$

(the last equation is true in a unitary space as well, since according to Proposition 3 the number  $\mu$  is real). Therefore, by virtue of self-adjointness,

$$\lambda (\mathbf{x}, \mathbf{y}) = \mu (\mathbf{x}, \mathbf{y}),$$

which is possible for  $\lambda \neq \mu$  only when  $(\mathbf{x}, \mathbf{y}) = 0$ .  $\square$

**Proposition 5 (on the orthogonal complement).** *For any self-adjoint operator  $A$ , the orthogonal complement  $\mathcal{F}^\perp$  of an arbitrary invariant subspace  $\mathcal{F}$  is also an invariant subspace.*

*Proof.* If  $\mathbf{x} \in \mathcal{F}^\perp$ , then  $(\mathbf{x}, \mathbf{y}) = 0$  for all  $\mathbf{y} \in \mathcal{F}$  and therefore  $(A\mathbf{x}, \mathbf{y}) = (\mathbf{x}, A\mathbf{y}) = 0$ , since by hypothesis  $A\mathbf{y} \in \mathcal{F}$ . Hence  $A\mathbf{x} \in \mathcal{F}^\perp$ .  $\square$

**Proposition 6 (on multiplicities).** *The geometric multiplicity  $p_{\lambda_0}$  of an arbitrary eigenvalue  $\lambda_0$  of a self-adjoint operator  $A$  equals its algebraic multiplicity  $n_{\lambda_0}$ :*

$$p_{\lambda_0} = n_{\lambda_0}.$$

*Proof.* Let  $\mathcal{F}_{\lambda_0}$  be a proper subspace belonging to an eigenvalue  $\lambda_0$  and let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be an orthonormal basis of a space  $\mathcal{V}$  such that  $\mathcal{F}_{\lambda_0} = [\mathbf{e}_1, \dots, \mathbf{e}_p]$  (and therefore such that  $\mathcal{F}_{\lambda_0}^\perp = [\mathbf{e}_{p+1}, \dots, \mathbf{e}_n]$ ), where  $p = p_{\lambda_0}$ . Since, according to Proposition 5,

$$\mathcal{F}_{\lambda_0} \oplus \mathcal{F}_{\lambda_0}^\perp = \mathcal{V},$$

in that basis the operator  $A$  has a matrix of the form

$$\left( \begin{array}{ccc|ccc} \lambda_0 & & & & & \\ & \ddots & & & & \\ & & \ddots & & & \\ & & & \lambda_0 & & \\ \hline & & & & 0 & \\ & & & & & \ddots \\ & & & & & & B \end{array} \right),$$

where  $B$  is the matrix of an operator  $B = A|_{\mathcal{P}_{\lambda_0}^\perp}$ . Hence  $f_A(\lambda) = (\lambda_0 - \lambda)^p f_B(\lambda)$  and therefore if  $p_{\lambda_0} < n_{\lambda_0}$ , then  $f_B(\lambda_0) = 0$  and so  $\lambda_0$  is an eigenvalue of the operator  $B$ . The corresponding eigenvector in  $\mathcal{P}_{\lambda_0}^\perp$  is an eigenvector of the operator  $A$  belonging to the eigenvalue  $\lambda_0$ , which is impossible since all these vectors lie in  $\mathcal{P}_{\lambda_0}$ . Consequently,  $p_{\lambda_0} \geq n_{\lambda_0}$  and hence  $p_{\lambda_0} = n_{\lambda_0}$  (since always  $p_{\lambda_0} \leq n_{\lambda_0}$ ; see Lecture 14).  $\square$

**Remark.** In the proof of Proposition 6 we used only the property of a self-adjoint operator, that the orthogonal complement of each of its subspaces is an invariant subspace (so we did not even need to fully use Proposition 5). Therefore *Proposition 6 is true for any operator for which the orthogonal complement of every proper subspace is invariant.*  $\square$

According to Theorem 1 of Lecture 16, it follows from Proposition 6 (together with Proposition 3, for Euclidean spaces) that an operator  $A$  is diagonalizable, i.e.

$$\mathcal{V} = \mathcal{P}_{\lambda_1} \oplus \dots \oplus \mathcal{P}_{\lambda_m},$$

where  $\lambda_1, \dots, \lambda_m$  are all possible eigenvalues of that operator. By choosing an orthonormal basis in each of the subspaces  $\mathcal{P}_{\lambda_i}$  we obtain, in view of Proposition 4, an orthonormal basis of a space  $\mathcal{V}$  in which the operator  $A$  has a diagonal matrix.

**Definition 4.** An operator  $A$  in a Euclidean or unitary space  $\mathcal{V}$  is said to be *orthogonally diagonalizable* if in the space  $\mathcal{V}$  there exists an orthonormal basis in which the matrix of the operator  $A$  is diagonal (i.e. which consists of eigenvectors of that operator).

We thus see that we have proved the following theorem.

**Theorem 1.** *In a Euclidean or unitary space, any self-adjoint operator is orthogonally diagonalizable.*  $\square$

# Lecture 20

*Bringing quadratic forms into canonical form by orthogonal transformation of variables. Second degree hypersurfaces in a Euclidean point space. The minimax property of eigenvalues of self-adjoint operators. Orthogonally diagonalizable operators*

Theorem 1 of the preceding lecture states that in every Euclidean space any self-adjoint operator is orthogonally diagonalizable. We reformulate the theorem in terms of symmetric bilinear (or, equivalently, quadratic) forms.

Let

$$(1) \quad Q(x_1, \dots, x_n) = \sum_{i,j=1}^n q_{ij} x_i x_j$$

be an arbitrary quadratic form in  $n$  variables  $x_1, \dots, x_n$  with real coefficients  $q_{ij}$ ,  $i, j = 1, \dots, n$ .

On choosing in an  $n$ -dimensional Euclidean space  $\mathcal{V}$  an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  we may consider in  $\mathcal{V}$  a quadratic functional  $Q$  expressed in that basis as  $Q(x_1, \dots, x_n)$  and hence the corresponding symmetric linear operator  $Q: \mathcal{V} \rightarrow \mathcal{V}$  (i.e. such that  $Q(\mathbf{x}) = (Q\mathbf{x}, \mathbf{x})$  for any vector  $\mathbf{x} \in \mathcal{V}$ ). According to Theorem 1 of the preceding lecture, in the space  $\mathcal{V}$  there exists an orthonormal basis  $\mathbf{f}_1, \dots, \mathbf{f}_n$  in which the operator  $Q$  has a diagonal matrix with diagonal elements  $\lambda_1, \dots, \lambda_n$ . This implies that for any vector  $\mathbf{x} \in \mathcal{V}$  we have

$$Q(\mathbf{x}) = \lambda_1 y_1^2 + \dots + \lambda_n y_n^2,$$

where

[illegible]

are the coordinates of the vector  $\mathbf{x}$  in the basis  $\mathbf{f}_1, \dots, \mathbf{f}_n$ . Since both bases  $\mathbf{e}_1, \dots, \mathbf{e}_n$  and  $\mathbf{f}_1, \dots, \mathbf{f}_n$  are orthonormal, transformation (2) is orthogonal, i.e. the matrix  $C$  of its coefficients is an orthogonal matrix (see Lecture 14 in [1]). This proves the following theorem.

**Theorem 1.** *Any quadratic form (1) can be reduced by the orthogonal transformation of the variables to the form*

$$(3) \quad \lambda_1 y_1^2 + \dots + \lambda_n y_n^2.$$

The coefficients  $\lambda_1, \dots, \lambda_n$  are the roots of the equation

$$\det (Q - \lambda E) = 0$$

and are therefore uniquely determined (up to an order).  $\square$

The theorem formally differs from the (substantially simpler) Lagrange theorem of Lecture 11 only in that bringing into the canonical form (3) is achieved not by an arbitrary, but by the orthogonal transformation of the variable (2). That is why the canonical form (3) proves to be unique.

Just as the Lagrange theorem allowed us to give a classification of second degree hypersurfaces of an  $n$ -dimensional affine space (see Lecture 12), so Theorem 1 leads to a similar classification of second degree hypersurfaces in an  $n$ -dimensional Euclidean point (real-complex) space. Indeed, repeating word for word the proof of Theorem 5 in Lecture 13 and only referring to Theorem 1 instead of the Lagrange theorem we obtain immediately the following theorem.

**Theorem [2 (bringing the equations of second degree hypersurfaces in an  $n$ -dimensional Euclidean space into canonical form).]** *For any second degree hypersurface in an  $n$ -dimensional ( $n \geq 1$ ) real-complex Euclidean space there exists a system of rectangular coordinates  $x_1, \dots, x_n$  in which its equation has either the form*

$$(I) \quad \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = \varepsilon,$$

where  $1 \leq r \leq n$  and  $\varepsilon = 0$  or  $1$ , or (which is possible only when  $n > 1$ ) the form

$$(II) \quad \lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = 2x_{r+1},$$

where  $1 \leq r \leq n - 1$ , with  $\lambda_1 \neq 0, \dots, \lambda_r \neq 0$  in both cases.  $\square$

In order to uniquely fix the coefficients  $\lambda_1, \dots, \lambda_r$  (which, we note, are proportional to the nonzero roots of the corresponding characteristic polynomial, repeated as many times as is their multiplicity) one should first order them in a reasonable way (i.e. interchange appropriately the coordinates  $x^1, \dots, x^n$ ). We require that first the positive coefficients should be transferred and then the negative ones. Besides, in either group the coefficients should be arranged in the order of increasing absolute values. Thus, if  $p$ ,  $0 \leq p \leq r$ , is the number of positive coefficients, then we assume that

$$0 < \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_p$$

and

$$0 < |\lambda_{p+1}| \leq |\lambda_{p+2}| \leq \dots \leq |\lambda_r|.$$

We can in addition get

$$(4) \quad 0 \leq p \leq \left[ \frac{r}{2} \right]$$

for  $\varepsilon = 0$  in case (I) by multiplying by  $-1$ . We can obtain the same result also in case (II) by changing, if necessary, the sign of the coordinate  $x_{r+1}$ . Therefore, for the purpose of uniformity, we shall assume in case (I) the value  $\varepsilon = -1$ , satisfying in this way condition (4).

Finally, we shall suppose in case (I) for  $\varepsilon = 0$  that

$$|\lambda_1| + \dots + |\lambda_r| = 1.$$

Equations (I) and (II) satisfying these conditions will be called the *Euclidean canonical equations* of second degree hypersurfaces.

For  $n = 2$  and  $n = 3$  we obviously obtain (up to notation) the canonical equations of second degree curves in the plane and of second degree surfaces in three-dimensional space, enumerated in Lectures 22 and 23 of [1].

Bringing the equations of hypersurfaces into canonical form by the method employed in proving Theorem 2 (i.e. by the method of Lecture 13 making use of Theorem 1 instead of the Lagrange theorem) we shall all the time obtain, as can easily be seen, the same canonical equation (although, possibly, in different systems of coordinates). Although this does not prove yet that there are no coordinates in



which one obtains a different canonical equation, nevertheless it is so:

**Theorem 3 (classification of second degree hypersurfaces of an  $n$ -dimensional real-complex Euclidean space).** *Two second degree hypersurfaces in an  $n$ -dimensional real-complex Euclidean space are Euclidean equivalents if and only if they have the same canonical equations.*

We know, from the example of second degree curves in the plane (see Lecture 22 in [1]), how to proceed in proving this theorem. The method is to characterize the coefficients  $\lambda_1, \dots, \lambda_r$  geometrically regardless of coordinates. To clarify the idea of the general method, let us consider in the plane an ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where  $a \geq b$  (in this case,  $\lambda_1 = \frac{1}{a^2}$ ,  $\lambda_2 = \frac{1}{b^2}$ ).

The left-hand side  $\frac{x^2}{a^2} + \frac{y^2}{b^2}$  is a quadratic form in the coordinates  $x, y$  of the points of the plane. If we consider this quadratic form only for  $x^2 + y^2 = 1$  (on a “unit circle”), then, as can easily be seen, its maximum is  $\lambda_2 = \frac{1}{b^2}$  and its minimum is  $\lambda_1 = \frac{1}{a^2}$ .

In the case of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad a \geq b \geq c,$$

the coefficient  $\frac{1}{c^2}$  is similarly equal to the maximum of the quadratic form  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$  on a “unit sphere”  $x^2 + y^2 + z^2 = 1$ , and the coefficient  $\frac{1}{a^2}$  is equal to the minimum.

The “middle” coefficient  $\frac{1}{b^2}$  is more difficult to characterize. To this end consider all possible sections of an ellipsoid by planes passing through its centre. These sections are ellipses and the corresponding coefficients  $\lambda_1, \lambda_2 \geq \lambda_1$  are defined for them. These coefficients are of course dependent on the

choice of the plane and, as can easily be seen, *the lowest possible value of the largest coefficient*  $\lambda_2$  is just equal to  $\frac{1}{b^2}$ .

It turns out that a similar geometrical characterization of the coefficients  $\lambda_1, \dots, \lambda_r$  is possible in the general case as well. This is based on the corresponding statement about the eigenvalues of operators and we shall restrict ourselves to the proof of that statement. The transition to the coefficients of the equations of hypersurfaces is quite trivial, but we have no time to spare.

So we again return to the Euclidean vector space  $\mathcal{V}$  and the symmetric operator  $A$  given in it. We may assume, however, without any changes in the formulations and proof that the space  $\mathcal{V}$  is unitary and the operator  $A$  is Hermitian.

In both cases (see Proposition 3 of the preceding lecture) all eigenvalues (= characteristic roots) of the operator  $A$  are real. By repeating each of them as many times as is its multiplicity (and hence obtaining precisely  $n$  of them) we number the eigenvalues in decreasing order:

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n.$$

Our aim is to find a direct “geometrical” description of these numbers.

Let  $\mathcal{P}$  be an arbitrary subspace of the space  $\mathcal{V}$  and  $S = S(\mathcal{P})$ , its subset (“unit sphere”) consisting of all vectors  $\mathbf{x} \in \mathcal{P}$  for which  $(\mathbf{x}, \mathbf{x}) = 1$ .

Since for any vector  $\mathbf{x} \in S$  the number  $(A\mathbf{x}, \mathbf{x})$  is real (when  $\mathcal{V}$  is Euclidean, this is self-evident, and when  $\mathcal{V}$  is unitary, it is ensured by Proposition 2 of the preceding lecture), the number

$$\alpha(\mathcal{P}) = \sup \{ (A\mathbf{x}, \mathbf{x}); \quad \mathbf{x} \in \mathcal{P}, (\mathbf{x}, \mathbf{x}) = 1 \}$$

is defined (instead of sup one may write max, however, since the sphere  $S(\mathcal{P})$  is compact).

**Proposition 1.** *For any  $q = 1, \dots, n$ , we have*

$$\lambda_q = \inf \{ \alpha(\mathcal{P}); \quad \dim \mathcal{P} = n - q + 1 \},$$

where inf is taken over all subspaces  $\mathcal{P} \subset \mathcal{V}$  of dimension  $n - q + 1$ .

*Proof.* In the space  $\mathcal{V}$ , according to Theorem 1 of the preceding lecture, there exists an orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  such that

$$A\mathbf{e}_q = \lambda_q \mathbf{e}_q \text{ for any } q = 1, \dots, n.$$

Let  $\mathcal{P}_q = [\mathbf{e}_1, \dots, \mathbf{e}_q]$  and let  $\mathcal{P}$  be an arbitrary subspace of dimension  $n - q + 1$ . Since

$$\dim \mathcal{P}_q + \dim \mathcal{P} = q + (n - q + 1) = n + 1 > n,$$

we have, according to Theorem 1 of Lecture 1,  $\mathcal{P}_q \cap \mathcal{P} \neq \{0\}$ , i.e. there exists a nonzero vector  $\mathbf{x} \in \mathcal{P}_q \cap \mathcal{P}$ . We may assume without loss of generality that  $(\mathbf{x}, \mathbf{x}) = 1$ . Since  $\mathbf{x} \in \mathcal{P}$ , we have  $\alpha(\mathcal{P}) \geq (A\mathbf{x}, \mathbf{x})$ , and since  $\mathbf{x} \in \mathcal{P}_q$  and hence  $\mathbf{x} = x_1\mathbf{e}_1 + \dots + x_q\mathbf{e}_q$ , we have

$$\begin{aligned} (A\mathbf{x}, \mathbf{x}) &= (\lambda_1 x_1 \mathbf{e}_1 + \dots + \lambda_q x_q \mathbf{e}_q, x_1 \mathbf{e}_1 + \dots + x_q \mathbf{e}_q) = \\ &= \lambda_1 |x_1|^2 + \dots + \lambda_q |x_q|^2 \geq \lambda_q (|x_1|^2 + \dots + |x_q|^2) = \\ &= \lambda_q (\mathbf{x}, \mathbf{x}) = \lambda_q. \end{aligned}$$

Thus  $\alpha(\mathcal{P}) \geq \lambda_q$  for any subspace  $\mathcal{P}$  of dimension  $n - q + 1$  and hence

$$\inf \{ \alpha(\mathcal{P}); \dim \mathcal{P} = n - q + 1 \} \geq \lambda_q.$$

On the other hand, since for any vector  $\mathbf{x} = x_q \mathbf{e}_q + \dots + x_n \mathbf{e}_n$  of the subspace  $\mathcal{P}_{(q)} = [\mathbf{e}_q, \dots, \mathbf{e}_n]$  of dimension  $n - q + 1$  there is an inequality

$$\begin{aligned} (A\mathbf{x}, \mathbf{x}) &= \lambda_q |x_q|^2 + \dots + \lambda_n |x_n|^2 \leq \\ &\leq \lambda_q (|x_q|^2 + \dots + |x_n|^2) = \lambda_q (\mathbf{x}, \mathbf{x}) = \lambda_q, \end{aligned}$$

we have

$$\alpha(\mathcal{P}_{(q)}) \leq \lambda_q,$$

and hence

$$\inf \{ \alpha(\mathcal{P}); \dim \mathcal{P} = n - q + 1 \} \leq \lambda_q. \quad \square$$

The property of the eigenvalues of self-adjoint operators we have proved is called the *minimax property of eigenvalues*.

The proof of Theorem 3 is now obvious. We leave it to the reader to give the details.

In a Euclidean space every orthogonally diagonalizable operator, having in some orthonormal basis a diagonal, and hence symmetric, matrix, is symmetric (self-adjoint). This proves the following theorem.

**Theorem 4.** *In a Euclidean space a linear operator is orthogonally diagonalizable if and only if it is symmetric.  $\square$*

In a unitary space, however, self-adjoint (Hermitian) operators make only a part of all orthogonally diagonalizable operators, since in a Hermitian matrix all diagonal elements must be real. Therefore an operator having in some orthonormal basis a diagonal matrix at least one of whose elements is nonreal is orthogonally diagonalizable but not Hermitian.

**Definition 1.** An operator  $A$  in a unitary (or Euclidean) space is said to be *normal*, if it is commutative with the adjoint operator  $A^*$ .

Recall (see the preceding lecture) that in a unitary space any operator  $A$  can be uniquely represented as

$$A = B + iC,$$

where  $B$  and  $C$  are Hermitian operators.

**Proposition 2.** *In a unitary space an operator  $A = B + iC$  is normal if and only if the operators  $B$  and  $C$  are commutative ( $BC = CB$ ).*

*Proof.* Since

$$A^* = B^* + (iC)^* = B^* - iC^* = B - iC,$$

we have

$$AA^* = (B + iC)(B - iC) = B^2 + C^2 + i(CB - BC)$$

and

$$A^*A = (B - iC)(B + iC) = B^2 + C^2 - i(CB - BC).$$

Therefore  $AA^* = A^*A$  if and only if  $CB - BC = 0$ .  $\square$

Note that for a normal operator  $A$  the operator  $AA^* = A^*A$  is expressed by the formula

$$AA^* = B^2 + C^2$$

similar to the formula for the square of the modulus of a complex number.

If in some orthonormal basis an operator  $A$  has a diagonal matrix  $A$ , then in the same basis the adjoint operator has a complex conjugate and transposed, and hence also diagonal, matrix. Since any two diagonal matrices commute, so do the operators  $A$  and  $A^*$ . This proves that *in a unitary space any orthogonally diagonalizable operator is normal*.  $\square$

Our immediate aim is to prove the converse. To do this we shall try to extend to the case of normal operators Propositions 3 to 5 of the preceding lecture.

Proposition 3 of Lecture 19 cannot of course be directly generalized to normal operators, since the eigenvalues (= characteristic roots) of a normal operator may be any complex numbers. Its analogue for normal operators is the following proposition from which incidentally Proposition 3 of Lecture 19 immediately follows for unitary spaces:

**Proposition 3.** *Any eigenvector of a normal operator  $A$  belonging to an eigenvalue  $\lambda$  is an eigenvector of the adjoint operator  $A^*$  belonging to an eigenvalue  $\bar{\lambda}$ .*

*Proof.* If the operator  $A$  is normal, then for any vector  $x$

$$(Ax, Ax) = (A^*Ax, x) = (AA^*x, x) = (A^*x, A^*x),$$

i.e.

$$|Ax| = |A^*x|.$$

Since every operator of the form  $A - \lambda E$  is normal, as well as the operator  $A$ , it follows (as  $(A - \lambda E)^* = A^* - \bar{\lambda}E$ ) that for any  $\lambda$

$$|(A - \lambda E)x| = |(A^* - \bar{\lambda}E)x|.$$

Therefore, if  $(A - \lambda E)x = 0$ , then  $(A^* - \bar{\lambda}E)x = 0$ .  $\square$

Proposition 4 of Lecture 19 remains unaffected for normal operators:

**Proposition 4.** *Any two eigenvectors  $x$  and  $y$  of a normal operator  $A$  belonging to different eigenvalues  $\lambda$  and  $\mu$  are orthogonal.*

*Proof.* If  $Ax = \lambda x$ , then  $(Ax, y) = \lambda (x, y)$ . Similarly, if  $Ay = \mu y$  and hence, according to Proposition 3,  $A^*y = \bar{\mu}y$ , then  $(x, A^*y) = (x, \bar{\mu}y) = \mu (x, y)$ . Consequently,  $\lambda (x, y) = (Ax, y) = (x, A^*y) = \mu (x, y)$  and therefore  $(x, y) = 0$  (for by hypothesis  $\lambda \neq \mu$ ).  $\square$

On the contrary, Proposition 5 of Lecture 19 is in general false for normal operators: there exist normal operators having invariant subspaces with noninvariant orthogonal complement (construct an example!). For proper subspaces it proves to be true, however:

**Proposition 5.** *The orthogonal complement  $\mathcal{P}_\lambda^\perp$  of an arbitrary proper subspace  $\mathcal{P}_\lambda$  of a normal operator  $A$  is invariant under  $A$ .*

*Proof.* If  $x \in \mathcal{P}_\lambda^\perp$ , then  $(x, y) = 0$  for any vector  $y \in \mathcal{P}_\lambda$ . Therefore  $(Ax, y) = (x, A^*y) = (x, \bar{\lambda}y) = \bar{\lambda}(x, y) = 0$ , for according to Proposition 3  $A^*y = \bar{\lambda}y$ . Consequently  $Ax \in \mathcal{P}_\lambda^\perp$ .  $\square$

As was already noted in the preceding lecture, it is only this property of the operator  $A$  that is necessary in the proof of Proposition 6. Therefore this proposition remains valid for any normal operator, which, in view of Proposition 4, ensures the orthogonal diagonalizability of the operator.

We have thus proved the following theorem:

**Theorem 5.** *In a unitary space a linear operator is orthogonally diagonalizable if and only if it is normal.*  $\square$

This theorem allows the properties of a normal operator to be reduced to those of its spectrum. For example, it is now obvious that *in a unitary space a normal operator  $A$  is*

- (a) *Hermitian,*
- (b) *invertible,*
- (c) *idempotent (i.e.  $A^2 = A$ )*

*if and only if its eigenvalues are respectively*

- (a') *real,*
- (b') *nonzero,*
- (c') *equal to zero or unity.*

Note that the implications  $(a) \Rightarrow (a')$ ,  $(b) \Rightarrow (b')$ , and  $(c) \Rightarrow (c')$  hold for any linear operators. The inverse—most interesting—implications, however, hold only for normal operators (construct corresponding examples!).

Of course, similar statements about the equivalence of the properties hold also for symmetric operators in a Euclidean space.

# Lecture 21

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*Positive operators • Isometric operators • Unitary matrices • Polar factorization of invertible operators • A geometrical interpretation of polar factorization • Parallel translations and centroaffine transformations • Bringing a unitary operator into diagonal form • A rotation of an  $n$ -dimensional Euclidean space as a composition of rotations in two-dimensional planes*

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**Proposition 1.** *The following properties of a linear operator  $A$  are equivalent, in a Euclidean or a unitary space  $\mathcal{V}$ :*

(a) *There exists a self-adjoint operator  $B$  such that*

$$A = B^2.$$

(b) *There exists a linear operator  $C$  such that*

$$A = C^*C.$$

(c) *The operator  $A$  is self-adjoint and  $(Ax, x) \geq 0$  for any vector  $x \in \mathcal{V}$ .*

(d) *The operator  $A$  is self-adjoint and all of its eigenvalues are nonnegative.*

*Also equivalent are the strengthened variants of these properties resulting when we require in (a) and (b) that the operators  $B$  and  $C$  should be invertible, in (c) that  $(Ax, x) > 0$  for  $x \neq 0$ , and in (d) that all eigenvalues should be positive.*

*Proof. Implication (a)  $\Rightarrow$  (b). It suffices to put  $C = B$ .*

*Implication (b)  $\Rightarrow$  (c). If  $A = C^*C$ , then  $(Ax, x) = (Cx, Cx) = |Cx|^2 \geq 0$ . Moreover, if the operator  $C$  is invertible and hence  $Cx \neq 0$  for  $x \neq 0$ , then  $(Ax, x) > 0$  for  $x \neq 0$ .*

*Implication (c)  $\Rightarrow$  (d).* If  $A\mathbf{x} = \lambda\mathbf{x}$ , then  $(A\mathbf{x}, \mathbf{x}) = \lambda (\mathbf{x}, \mathbf{x})$ , and therefore if  $(A\mathbf{x}, \mathbf{x})$  is nonnegative (positive), then  $\lambda$  is nonnegative (positive).

*Implication (d)  $\Rightarrow$  (a).* Let  $\mathbf{e}_1, \dots, \mathbf{e}_n$  be a basis consisting of eigenvectors of the operator  $A$  and let  $\lambda_1, \dots, \lambda_n$  be the corresponding eigenvalues. Since under the hypothesis  $\lambda_1 \geq 0, \dots, \lambda_n \geq 0$ , then there exist roots  $\sqrt{\lambda_1}, \dots, \sqrt{\lambda_n}$  (in  $\mathbb{R}$ ). We define the operator  $B$  by the formulas

$$(1) \quad B\mathbf{e}_1 = \sqrt{\lambda_1}\mathbf{e}_1, \dots, B\mathbf{e}_n = \sqrt{\lambda_n}\mathbf{e}_n.$$

It is clear that  $B^2 = A$ .  $\square$

**Definition 1.** An operator  $A$  is said to be *nonnegative* if it possesses properties (a) to (d). If the operator  $A$  possesses the strengthened properties (a) to (d), it is said to be *positive*. Every self-adjoint operator  $B$  satisfying the relation  $B^2 = A$  is called a *square root* of the operator  $A$ . A nonnegative (positive) square root is designated  $\sqrt{A}$ .

Formula (1) shows that *there does exist an operator  $\sqrt{A}$  and that it is uniquely defined for any nonnegative (positive) operator  $A$ .*  $\square$

It is obvious that a *nonnegative operator is positive if and only if it is invertible.*  $\square$

In a Euclidean space a self-adjoint operator  $A$  is positive if and only if a square functional  $(A\mathbf{x}, \mathbf{x})$  is positive definite.

Note that in a number of textbooks and monographs nonnegative operators are called *positive*, while positive operators are called *strictly positive*.

Positive operators are the analogues of positive real numbers. Now let us consider operators that are the analogues of complex numbers whose modulus is equal to unity.

**Proposition 2.** *The following properties of a linear operator  $A$  are equivalent, in a Euclidean or a unitary space  $\mathcal{V}$ :*

(a) *For any two vectors  $\mathbf{x}, \mathbf{y} \in \mathcal{V}$  we have*

$$(A\mathbf{x}, A\mathbf{y}) = (\mathbf{x}, \mathbf{y}).$$

(b) *For any vector  $\mathbf{x} \in \mathcal{V}$  we have*

$$|A\mathbf{x}| = |\mathbf{x}|.$$



(c) For any orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  the vectors  $\mathbf{A}\mathbf{e}_1, \dots, \mathbf{A}\mathbf{e}_n$  also constitute an orthonormal basis of that space.

(d) For the elements  $a_i^j$  of the matrix of an operator  $\mathbf{A}$ , in an arbitrary orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  there are relations

$$(2) \quad \sum_{k=1}^n a_i^k a_j^k = \delta_{ij}, \quad i, j = 1, \dots, n,$$

if the space  $\mathcal{V}$  is Euclidean and relations

$$(3) \quad \sum_{k=1}^n a_i^k \bar{a}_j^k = \delta_{ij}, \quad i, j = 1, \dots, n,$$

if the space  $\mathcal{V}$  is unitary.

(e) We have

$$\mathbf{A}^* \mathbf{A} = \mathbf{E}.$$

(f) The operator  $\mathbf{A}$  is invertible and

$$\mathbf{A}^{-1} = \mathbf{A}^*.$$

(g) We have

$$\mathbf{A} \mathbf{A}^* = \mathbf{E}.$$

(h) For the elements  $a_i^j$  of the matrix of an operator  $\mathbf{A}$ , in an arbitrary orthonormal basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of a space  $\mathcal{V}$  there are relations

$$(4) \quad \sum_{k=1}^n a_k^i a_k^j = \delta_{ij}, \quad i, j = 1, \dots, n,$$

if the space  $\mathcal{V}$  is Euclidean and relations

$$(5) \quad \sum_{k=1}^n a_k^i \bar{a}_k^j = \delta_{ij}, \quad i, j = 1, \dots, n,$$

if the space  $\mathcal{V}$  is unitary.

*Proof.* We shall prove that the following implications hold:

$$\begin{array}{c}
 (a) \iff (e) \iff (f) \iff (g) \iff (h) \\
 \begin{array}{c} \Downarrow \quad \nearrow \\ (b) \end{array} \\
 \begin{array}{c} \Downarrow \quad \Uparrow \\ (c) \iff (d) \end{array}
 \end{array}$$

*Implication*  $(a) \Rightarrow (b)$ . It suffices to put  $y = x$ .

*Implication*  $(a) \Rightarrow (c)$ . Since  $(Ae_i, Ae_j) = (e_i, e_j)$ , we have  $(Ae_i, Ae_j) = \delta_{ij}$ , if  $(e_i, e_j) = \delta_{ij}$ .

*Implication*  $(b) \Rightarrow (e)$ . If  $|Ax| = |x|$ , then  $((A^*A - E)x, x) = (A^*Ax, x) - (x, x) - (Ax, Ax) - (x, x) = |Ax|^2 - |x|^2 = 0$  and hence  $A^*A = E$  (in a Euclidean space  $\mathcal{V}$ , because the operator  $A^*A - E$  is symmetric, and in a unitary space  $\mathcal{V}$ , by Proposition 1 of Lecture 18).

*Implications*  $(c) \iff (d)$ . By definition  $Ae_i = a_i^j e_j$ . Therefore

$$(Ae_i, Ae_j) = \sum_{k=1}^n a_i^k a_j^k$$

in a Euclidean space and

$$(Ae_i, Ae_j) = \sum_{k=1}^n a_i^k \bar{a}_j^k$$

in a unitary space. Hence  $(c) \Rightarrow (d)$  and  $(d) \Rightarrow (c)$ .

*Implications*  $(a) \iff (e)$ . By definition  $(A^*Ax, y) = (Ax, Ay)$ . Therefore  $(a) \Rightarrow (e)$  and  $(e) \Rightarrow (a)$  (since for some operator  $C$  and any vectors  $x$  and  $y$  we have  $(Cx, y) = (x, y)$  if and only if  $C = E$ ).

*Implications*  $(d) \iff (e)$  and  $(g) \iff (h)$ . An operator  $A^*$  has a matrix  $(\bar{a}_i^j)$  in a basis  $e_1, \dots, e_n$ . Hence elements of the matrix of the operator  $AA^*$  are the sums  $\sum_k a_k^i \bar{a}_k^j$  and elements of the matrix of the operator  $A^*A$  are the sums  $\sum_k a_i^k \bar{a}_j^k$ . Therefore  $(d) \iff (e)$  and  $(g) \iff (h)$ .

*Implications*  $(e) \Rightarrow (f)$  and  $(g) \Rightarrow (f)$ . See implications  $1^\circ \Rightarrow 5^\circ$  and  $3^\circ \Rightarrow 5^\circ$  of Proposition 2 in Lecture 14.

Implications  $(f) \Rightarrow (e)$  and  $(f) \Rightarrow (g)$ . Hold by definition.  $\square$

**Definition 2.** In a Euclidean or a unitary space  $\mathcal{V}$  a linear operator  $A$  is said to be *isometric* if it possesses properties (a) to (h). Isometric operators are also called *orthogonal* in a Euclidean space  $\mathcal{V}$ , and *unitary* in a unitary space  $\mathcal{V}$ .

Property (a) implies that an operator  $A$  preserves scalar products (and hence, in particular, also angles), i.e. is a *homomorphism* (in fact, by virtue of (f), even an isomorphism) of a space  $\mathcal{V}$  onto itself.

Note that *any isometric operator is normal* ( $A^*A = AA^*$ ).  $\square$

As we know (Proposition 4 of Lecture 14 in [1]), real matrices possessing properties (2) or (4) are exactly *orthogonal matrices*. By analogy matrices with complex coefficients possessing properties (3) and (5) are *unitary matrices*. For these, the following analogue of Proposition 4 of Lecture 14 in [1] holds (the symbol  $\bar{A}^\top$  designates a transposed matrix all the elements of which have been replaced by complex conjugate numbers).

**Proposition 3.** A matrix  $A = (a_i^j)$  of order  $n$ , with complex coefficients, is unitary if and only if it has one (and hence all) of the following properties:

(a) The matrix  $A$  is a transition matrix connecting two orthonormal bases of an  $n$ -dimensional unitary space.

(b) The columns of the matrix  $A$  constitute an orthonormal family of vectors of a unitary space  $\mathbb{C}^n$ .

(c) We have

$$\bar{A}^\top A = E.$$

(d) The matrix  $A$  is invertible and

$$A^{-1} = \bar{A}^\top.$$

(e) We have

$$A \bar{A}^\top = E.$$

(f) The rows of the matrix  $A$  constitute an orthonormal family of vectors of a unitary space  $\mathbb{C}^n$ .

*Proof.* Let us introduce a linear operator  $A$  that has a matrix  $A$  in some orthonormal basis. Then properties (a)

to (f) turn into properties (c) to (h) of the operator  $A$  in Proposition 2.  $\square$

Since  $\det \bar{A}^T = \overline{\det A}$ , it follows from properties (c) and (e) that

$$|\det A| = 1$$

for any unitary matrix  $A$ .

It is obvious that all unitary matrices of order  $n$  form a group. This is called a *unitary group* and designated  $U(n)$ . Its subgroup consisting of unimodular ( $\det A = 1$ ) matrices is designated  $SU(n)$ .

**Proposition 4.** *In a Euclidean (unitary) space any invertible operator  $A$  is uniquely decomposed as a product of an isometric operator  $U$  and a positive operator  $P$ :*

$$(6) \quad A = PU.$$

*Proof.* According to Proposition 1 an operator  $A^*A$  is positive and therefore there exists a positive square root

$$P = \sqrt{A^*A}$$

Let  $U = AP^{-1}$ . Then  $U^* = (P^*)^{-1}A^* = P^{-1}A^*$  (for the operator  $P$  is self-adjoint) and therefore  $U^*U = P^{-1}A^*AP^{-1} = P^{-1}P^2P^{-1} = E$ . Thus  $A = UP$ , where the operator  $U$  is isometric and the operator  $P$  is positive.

If  $UP = VQ$ , where  $U, V$  are isometric operators and  $P$  and  $Q$  are positive operators, then  $PU^* = QV^*$  and therefore

$$P^2 = PU^*UP = QV^*VQ = Q^2.$$

Hence (a positive square root is extracted uniquely)  $P = Q$  and therefore  $U = V$ . This proves that decomposition (6) is unique.  $\square$

Decomposition (6) is usually called the *polar factorization* of an operator  $A$ . It is similar to the decomposition  $re^{i\varphi} = r(\cos \varphi + i \sin \varphi)$  of an arbitrary complex number as a product of its modulus  $r$  and a number  $e^{i\varphi}$  equal in absolute value to unity.

Recall (see Lecture 26 of [1]) that an *affine transformation* of an affine space  $\mathcal{A}$  is its arbitrary automorphism, i.e.

a transformation defined by equating coordinates in two affine coordinate systems. If in the space  $\mathcal{A}$  an initial point  $O$  is chosen, then an arbitrary affine transformation carries a point with a radius vector  $\mathbf{x}$  over into a point with a radius vector of the form

$$(7) \quad \mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{b},$$

where  $\mathbf{A}$  is some invertible linear operator and  $\mathbf{b}$  is a fixed vector (this is but a different way of writing formula (2) of Lecture 27 in [1]).

Similarly, an *orthogonal transformation* of a Euclidean point space  $\mathcal{E}$  is its transformation defined by equating coordinates in two Euclidean (rectangular) coordinate systems. It can be written using the same formula (7) but now with an orthogonal operator  $\mathbf{A}$ .

By analogy we can introduce *unitary point spaces*  $\mathcal{E}$  as affine spaces into whose associated vector space the structure of a unitary vector space is introduced. Automorphisms of such spaces are *unitary transformations* that can be written using formula (7) with a unitary operator  $\mathbf{A}$ .

Since any Euclidean (or unitary) point space is, in particular, affine, it makes sense to speak of its affine transformations (7). To a polar factorization  $\mathbf{A} = \mathbf{U}\mathbf{P}$  of an operator  $\mathbf{A}$  there corresponds then a representation of an affine transformation (7) as a composition of an affine transformation

$$(8) \quad \mathbf{y} = \mathbf{P}\mathbf{x}$$

and an orthogonal (or unitary) transformation

$$\mathbf{y} = \mathbf{U}\mathbf{x} + \mathbf{b}.$$

In appropriately chosen rectangular coordinates transformation (8) can be written as

$$\begin{aligned} y_1 &= \lambda_1 x_1, \\ &\dots \dots \dots \\ y_n &= \lambda_n x_n, \end{aligned}$$

where  $\lambda_1 > 0, \dots, \lambda_n > 0$ , and hence is a composition of  $n$  compressions toward  $n$  mutually perpendicular hyperplanes. This proves that *any affine transformation of an  $n$ -dimensional Euclidean (unitary) point space is a composition of an*

orthogonal (unitary) transformation and  $n$  compressions toward  $n$  mutually perpendicular hyperplanes.  $\square$

For  $n = 2$  this statement makes the content of Proposition 1 in Lecture 27 of [1].

For  $A = E$  transformation (7) has the form

$$y = x + b$$

and is called a (*parallel*) *translation* to the vector  $b$ . For  $b = 0$  transformation (7) has the form

$$y = Ax$$

and is called a *centroaffine transformation*. It leaves fixed a point  $O$  called its *centre*. Any affine transformation is a composition of a translation and a centroaffine transformation.

We stress that transformation (7), with  $b \neq 0$ , may well be a centroaffine transformation (with centre other than  $O$ ). For this to happen, it is necessary and sufficient that there should exist a vector  $x_0$  (the radius vector of a centre) satisfying the relation

$$x_0 = Ax_0 + b,$$

i.e. such that  $(A - E)x_0 = b$ . In particular, this is necessarily so if the operator  $A - E$  is invertible, i.e. if the number 1 is not an eigenvalue of the operator  $A$ .

An orthogonal transformation that is a centroaffine one is called a *generalized rotation*. It is called simply a *rotation* if the orthogonal operator  $A$  is unimodular (orientation-preserving).

To get at least a primary idea of rotations we must study orthogonal operators in greater detail. To this end it would be convenient first to consider unitary operators.

**Proposition 5.** *The spectrum of an arbitrary unitary operator  $A$  lies, in the plane of a complex variable, on a unit circle, i.e. the absolute value of any characteristic root  $\lambda$  of a unitary operator is equal to unity:*

$$|\lambda| = 1.$$

*Proof.* Any characteristic root  $\lambda$  is an eigenvalue over the field  $\mathbb{C}$ , i.e. there exists a vector  $\mathbf{x}_0 \neq \mathbf{0}$  such that  $A\mathbf{x}_0 = \lambda\mathbf{x}_0$ . Then

$$(\mathbf{x}_0, \mathbf{x}_0) = (A\mathbf{x}_0, A\mathbf{x}_0) = (\lambda\mathbf{x}_0, \lambda\mathbf{x}_0) = \lambda\bar{\lambda}(\mathbf{x}_0, \mathbf{x}_0)$$

and hence  $\lambda\bar{\lambda} = 1$ .  $\square$

**Theorem 1.** *For any unitary operator  $A$  there exists an orthonormal basis in which the matrix of the operator  $A$  is of the form*

$$\begin{pmatrix} e^{i\varphi_1} & & 0 \\ & \ddots & \\ 0 & & e^{i\varphi_n} \end{pmatrix}.$$

*Proof.* A unitary operator is normal and hence orthogonally diagonalizable. This, together with Proposition 5, proves the theorem.  $\square$

Now let  $A$  be an orthogonal operator in a (real) Euclidean space  $\mathcal{V}$ .

We define its complexification

$$A^{\mathbb{C}}(\mathbf{x} + i\mathbf{y}) = A\mathbf{x} + iA\mathbf{y}$$

which is (see Lecture 17) a linear operator on the complexification

$$\mathcal{V}^{\mathbb{C}} = \mathcal{V} + i\mathcal{V}$$

of the space  $\mathcal{V}$ .

For any vectors

$$\mathbf{z} = \mathbf{x} + i\mathbf{y} \in \mathcal{V}^{\mathbb{C}}, \quad \mathbf{z}_1 = \mathbf{x}_1 + i\mathbf{y}_1 \in \mathcal{V}^{\mathbb{C}}$$

we set

$$(\mathbf{z}, \mathbf{z}_1) = [(\mathbf{x}, \mathbf{x}_1) + (\mathbf{y}, \mathbf{y}_1)] - i[(\mathbf{x}, \mathbf{y}_1) - (\mathbf{x}_1, \mathbf{y})].$$

A routine check shows that the functional  $\mathbf{z}, \mathbf{z}_1 \mapsto (\mathbf{z}, \mathbf{z}_1)$  is sesquilinear, Hermitian and positive definite, i.e. may be taken as a scalar multiplication in the complex vector space  $\mathcal{V}^{\mathbb{C}}$ . Under this multiplication the space  $\mathcal{V}^{\mathbb{C}}$  is thus a unitary space.

Further, since

$$\begin{aligned} (A^{\mathbb{C}}z, A^{\mathbb{C}}z_1) &= \\ &= [(Ax, Ax_1) + (Ay, Ay_1)] - i [(Ax, Ay_1) - (Ax_1, Ay)] = \\ &= [(x, x_1) + (y, y_1)] - i [(x, y_1) - (x_1, y)] = (z, z_1), \end{aligned}$$

the complexification  $A^{\mathbb{C}}$  of the orthogonal operator  $A$  is a unitary operator. Therefore, in particular, the operator  $A^{\mathbb{C}}$  is diagonalizable.

It follows (see Theorem 1 of Lecture 17) that in the space  $\mathcal{V}$  there exists a basis in which the matrix of the operator  $A$  is a direct sum of first order matrices of the form  $\lambda$  and second order matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}.$$

The real numbers  $\lambda$  are characteristic roots of the operator  $A^{\mathbb{C}}$  and therefore  $|\lambda| = 1$ , i.e.  $\lambda = \pm 1$ . As to the numbers  $\alpha, \beta$ , they are the real part and the coefficient of the imaginary part of the nonreal characteristic root  $\lambda = e^{i\varphi}$  of the operator  $A^{\mathbb{C}}$  and hence  $\alpha = \cos \varphi$  and  $\beta = \sin \varphi$ , where  $-\pi < \varphi \leq \pi$  and  $\varphi \neq 0$ .

Since the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

are also of the form

$$(9) \quad \begin{pmatrix} \cos \varphi & \sin \varphi \\ -\sin \varphi & \cos \varphi \end{pmatrix}$$

(for  $\varphi = 0$  and  $\varphi = \pi$  respectively), it follows that in some basis  $e_1, \dots, e_n$  of the space  $\mathcal{V}$  the matrix of an orthogonal operator is a direct sum of  $m$  matrices of the form (9) (with  $-\pi < \varphi \leq \pi$ ) and one first order matrix ( $\pm 1$ ) in the case  $n = 2m + 1$ , and either a direct sum of  $m$  matrices of the form (9) or a direct sum of  $m - 1$  of such matrices and a matrix of the form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in the case  $n = 2m$ .  $\square$



According to the construction described in Lecture 17, a basis  $\mathbf{e}_1, \dots, \mathbf{e}_n$  of the space  $\mathcal{V}$  is obtained from some basis  $\mathbf{e}_1^{\mathbb{C}}, \dots, \mathbf{e}_n^{\mathbb{C}}$  of the space  $\mathcal{V}^{\mathbb{C}}$  having the following two properties:

(a) every vector  $\mathbf{e}_q^{\mathbb{C}}$  is an eigenvector of the unitary operator  $\mathbf{A}^{\mathbb{C}}$ ;

(b) if an eigenvalue  $\lambda_q = e^{i\varphi_q}$  to which the eigenvector  $\mathbf{e}_q^{\mathbb{C}}$  belongs is real (i.e.  $\varphi_q = 0, \pi$ ), then so is the vector  $\mathbf{e}_q^{\mathbb{C}}$ , and if  $0 < \varphi_q < \pi$ , then the vector  $\mathbf{e}_{q+1}^{\mathbb{C}}$  is complex conjugate to the vector  $\mathbf{e}_q^{\mathbb{C}}$  belonging to the complex conjugate eigenvalue  $\bar{\lambda}_q = e^{-i\varphi_q}$ .

Also

$$\mathbf{e}_q^{\mathbb{C}} = \begin{cases} \mathbf{e}_q, & \text{if } \varphi_q = 0, \pi, \\ \mathbf{e}_q + i\mathbf{e}_{q+1} & \text{if } 0 < \varphi_q < \pi \\ \mathbf{e}_{q-1} - i\mathbf{e}_q & \text{if } -\pi < \varphi_q < 0 \end{cases}$$

Moreover, in addition to properties (a) and (b) we may assume the basis  $\mathbf{e}_1^{\mathbb{C}}, \dots, \mathbf{e}_n^{\mathbb{C}}$  to be orthonormal (since the operator  $\mathbf{A}^{\mathbb{C}}$  is diagonalizable orthogonally). Since

$$\mathbf{e}_q = \begin{cases} \mathbf{e}_q^{\mathbb{C}} & \text{when } \varphi_q = 0, \pi, \\ \frac{\mathbf{e}_q^{\mathbb{C}} + \mathbf{e}_{q+1}^{\mathbb{C}}}{2} & \text{when } 0 < \varphi_q < \pi, \\ \frac{\mathbf{e}_{q-1}^{\mathbb{C}} - \mathbf{e}_q^{\mathbb{C}}}{2i} & \text{when } -\pi < \varphi_q < 0, \end{cases}$$

the following equations hold

$$\begin{aligned} (\mathbf{e}_p, \mathbf{e}_q) &= 0, & \text{if } p \neq q, \\ (\mathbf{e}_p, \mathbf{e}_p) &= \begin{cases} 1 & \text{if } \varphi_p = 0, \pi, \\ 2 & \text{if } \varphi_p \neq 0, \pi. \end{cases} \end{aligned}$$

Consequently, if all vectors  $\mathbf{e}_p$  with  $\varphi_p \neq 0, \pi$  are multiplied by  $1/\sqrt{2}$ , an orthonormal basis results. Since, as is easily seen, the matrix of the operator  $\mathbf{A}$  remains unaltered under the operation, we have proved the following theorem.

**Theorem 2.** *For any orthogonal operator  $\mathbf{A}$ , in an  $n$ -dimensional Euclidean space  $\mathcal{V}$  there exists an orthonormal basis*

in which its matrix, for  $n = 2m + 1$ , has the form  
(10)

$$\left( \begin{array}{ccc} \varepsilon \begin{array}{|cc|} \hline \cos \varphi_1 & \sin \varphi_1 \\ -\sin \varphi_1 & \cos \varphi_1 \\ \hline \end{array} & & 0 \\ & \begin{array}{|cc|} \hline \cos \varphi_2 & \sin \varphi_2 \\ -\sin \varphi_2 & \cos \varphi_2 \\ \hline \end{array} & \\ & & \begin{array}{|cc|} \hline \cos \varphi_m & \sin \varphi_m \\ -\sin \varphi_m & \cos \varphi_m \\ \hline \end{array} \\ & 0 & \end{array} \right),$$

where  $\varepsilon = \pm 1$  and, for  $n = 2m$ , either the form

$$(11) \left( \begin{array}{ccc} \begin{array}{|cc|} \hline \cos \varphi_1 & \sin \varphi_1 \\ -\sin \varphi_1 & \cos \varphi_1 \\ \hline \end{array} & & 0 \\ & \begin{array}{|cc|} \hline \cos \varphi_2 & \sin \varphi_2 \\ -\sin \varphi_2 & \cos \varphi_2 \\ \hline \end{array} & \\ & & \begin{array}{|cc|} \hline \cos \varphi_m & \sin \varphi_m \\ -\sin \varphi_m & \cos \varphi_m \\ \hline \end{array} \\ & 0 & \end{array} \right),$$

or the form

$$(12) \left( \begin{array}{ccc} \begin{array}{|cc|} \hline \cos \varphi_1 & \sin \varphi_1 \\ -\sin \varphi_1 & \cos \varphi_1 \\ \hline \end{array} & & 0 \\ & \begin{array}{|cc|} \hline \cos \varphi_{m-1} & \sin \varphi_{m-1} \\ -\sin \varphi_{m-1} & \cos \varphi_{m-1} \\ \hline \end{array} & \\ & & \begin{array}{|cc|} \hline 1 & 0 \\ 0 & -1 \\ \hline \end{array} \\ & 0 & \end{array} \right). \quad \square$$

Note that the determinant of matrix (10) equals  $\varepsilon$ , the determinant of matrix (11) is positive (equals 1) and the determinant of matrix (12) is negative (equals  $-1$ ).

In terms of orthogonal transformations of point spaces Theorem 2 means that *any rotation of an  $n$ -dimensional Euclidean space is a composition of rotations in  $m = \left\lfloor \frac{n}{2} \right\rfloor$  mutually perpendicular two-dimensional planes and that any generalized orientation-reversing rotation is a composition of some rotation possessing an axis (i.e. a straight line all points of which remain fixed) and a reflection in a hyperplane perpendicular to that axis.* For  $n = 2m + 1$  any rotation possesses an axis, whereas for  $n = 2m$  there exist rotations without axes (these are rotations (11) for which  $\varphi_p \neq 0, \pi$ , with any  $p = 1, \dots, m$ ).

Since a rotation without axes (more precisely, the corresponding orthogonal operator in an associated vector space) has no eigenvalues equal to 1, its composition with any translation is again a rotation but with a different centre. A similar statement for rotations possessing axes is true if and only if the translation vector is parallel to none of the (many possible) axes of rotation. It follows that *any motion of a Euclidean space is a screw motion*, i.e. a composition of a rotation and a translation to a vector parallel to some rotation axis.

# Lecture 22

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*Smooth functions • Smooth hypersurfaces • Gradient • Derivatives with respect to a vector • Vector fields • Singular points of a vector field • A module of vector fields • Potential and irrotational vector fields • The rotation of a vector field • The divergence of a vector field • Vector analysis • Hamilton's symbolic vector • Formulas for products • Compositions of operators*

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The space  $\mathbb{R}^n$  of row vectors is not only a numerical model of  $n$ -dimensional affine or Euclidean spaces but also the domain of functions  $F(x_1, \dots, x_n)$  of  $n$  variables. Here geometry is closely interwoven with mathematical analysis (function theory) and becomes practically indistinguishable from it. It is no wonder therefore that one of the earliest, and at the same time one of the most important, rigorous definitions, or what is said to be *explications*, of the intuitive notion of a curve in the plane, of a surface in three-dimensional space and, in general, of a hypersurface in an  $n$ -dimensional space was given in analysis.

That definition proceeds from viewing a hypersurface (for  $n = 2$ , a curve) as a “locus” of points whose coordinates satisfy a condition of the form

$$(1) \quad F(x_1, \dots, x_n) = 0.$$

Since we want to explicate the notion of a “smooth” curve or a surface having no fractures, it is natural to assume the function  $F$  to be a *differentiable function of class  $C^\infty$* , i.e. a function having (automatically continuous) partial derivatives of all orders. It is usual, however, to use in practice (in proving theorems) mostly derivatives of the first and second orders and only seldom those of higher orders. There-

fore, in order not to violate the general-mathematical principle—not to introduce unessential propositions—we assume the function  $F$  to have continuous partial derivatives only up to some order  $k \geq 1$  inclusively. Moreover, in order to get rid of the irksome need to see to it that nowhere derivatives of higher orders should be used we shall not specify the order  $k$ , i.e. we shall simply require that all functions should have continuous partial derivatives of all the orders we shall need. For brevity we shall call such functions *smooth functions*.

The smoothness condition is of a local character and may fail at isolated points. To take this into account we shall consider equations of the form (1) not in the whole of  $\mathbb{R}^n$  but in some open set  $U \subset \mathbb{R}^n$  (for example, in an open ball). The set of all functions  $\mathbf{x} \mapsto F(\mathbf{x})$  defined and smooth at all points  $\mathbf{x} = (x_1, \dots, x_n) \in U$  will be designated  $\mathcal{F}(U)$ . It is obviously a ring and an (infinite-dimensional) vector space over the field  $\mathbb{R}$ .

For the simplest smooth functions (for example, polynomials) the sets given by condition (1) correspond quite well as a rule with the intuitive notion of surfaces, although often not in the entire space  $\mathbb{R}^n$  but only in some open set of the space. Therefore the opinion prevailed for a long time that the sets given by conditions of the form (1) with a smooth function  $F$  are more or less capable of pretending to the role of hypersurfaces (of curves, for  $n = 2$ ). And it came so much the more as a surprise when about forty years ago the American mathematician Whitney proved the theorem which states that *for any closed set  $C \subset \mathbb{R}^n$  there exists a smooth (class  $C^\infty$ ) function  $F$  in  $\mathbb{R}^n$  such that  $F(\mathbf{x}) = 0$  if and only if  $\mathbf{x} \in C$* . (It is easy to see that for the function  $F$  to exist it is necessary that the set  $C$  should be closed; it is a surprise that it is also sufficient that the set should be closed.) We shall prove the theorem in the third semester's lectures, and now we shall only give an example.

**Example.** The function  $F$  given by the formula

$$F(\mathbf{x}) = \begin{cases} 0 & \text{if } |\mathbf{x}| \leq 1; \\ \frac{1}{e^{|\mathbf{x}|-1}} & \text{if } |\mathbf{x}| > 1, \end{cases}$$

where  $|\mathbf{x}| = \sqrt{x_1^2 + \dots + x_n^2}$ , belongs to the class  $C^\infty$  in the whole of  $\mathbb{R}^n$ . Moreover, the set of all points  $\mathbf{x} \in \mathbb{R}^n$  for which  $F(\mathbf{x}) = 0$  is a ball (or a disk for  $n = 2$ )  $|\mathbf{x}| \leq 1$ .

Whitney's theorem explains why the condition of smoothness of the function  $F$  has to be supplemented with additional conditions. The *regularity condition* known from the course in analysis is that at any point of hypersurface (1) the vector

$$\text{grad } F = \left( \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n} \right)$$

(the so-called *gradient* of the function  $F$ ) should be nonzero, i.e. that at least one partial derivative

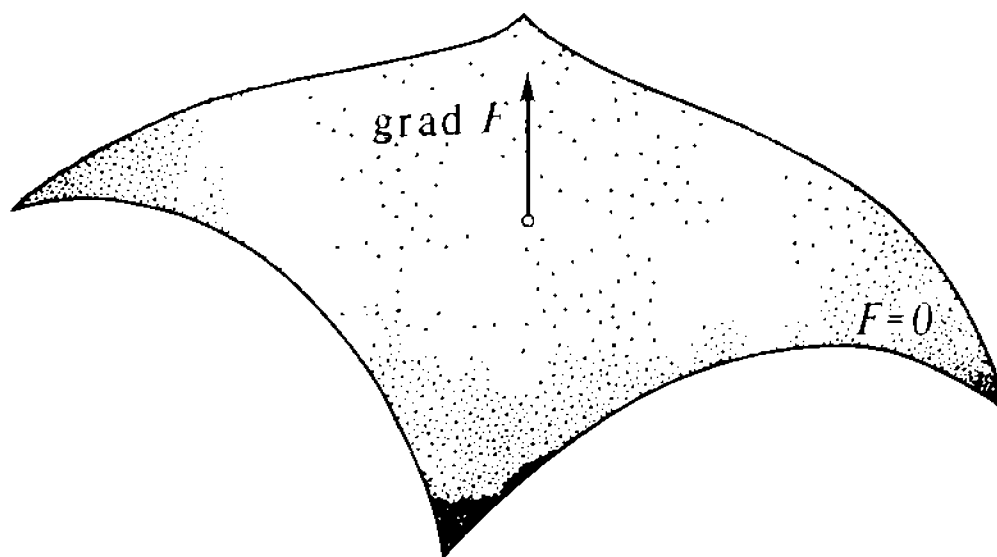
$$(2) \quad \frac{\partial F}{\partial x_1}, \dots, \frac{\partial F}{\partial x_n}$$

should be nonzero. Thus we arrive at the following definition.

**Definition 1.** A set  $\mathcal{H}$  of all points  $\mathbf{x} = (x_1, \dots, x_n)$  of an open set  $U \subset \mathbb{R}^n$  that satisfy the equation

$$(3) \quad F(\mathbf{x}) = 0,$$

where  $F$  is a function smooth in  $U$ , is said to be a *smooth* (or *regular*) *hypersurface* in  $U$  if at every point  $\mathbf{x} \in \mathcal{H}$  at least one partial derivative (2) is nonzero.

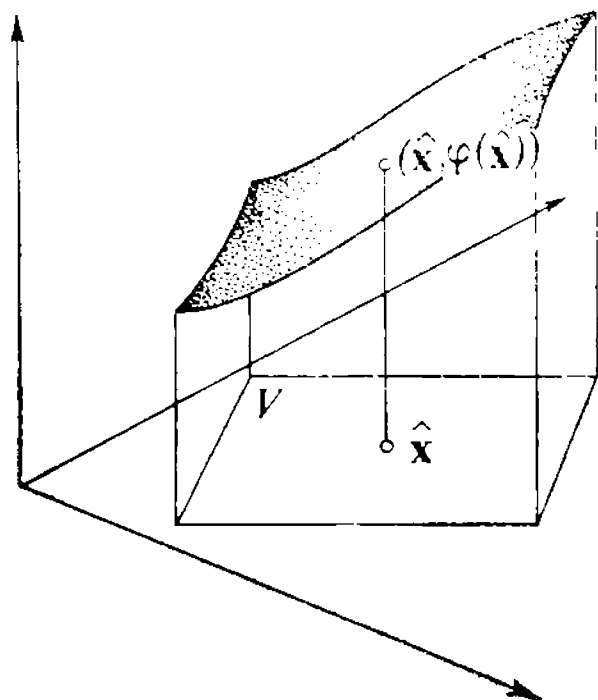


Points in the space  $\mathbb{R}^{n-1}$  will be designated by symbols of the form:  $\hat{\mathbf{x}}, \mathbf{y}, \dots$  and so on. And for any point  $\mathbf{x} = (x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n$  the symbol  $\hat{\mathbf{x}}$  will designate

a point  $(x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$ . Accordingly for any set  $C \subset \mathbb{R}^n$  the symbol  $\hat{C}$  will designate the set of all points  $\hat{\mathbf{x}} \in \mathbb{R}^{n-1}$ , where  $\mathbf{x} \in C$ . Instead of  $\mathbf{x} = (x_1, \dots, x_{n-1}, x_n)$  we shall also write  $\mathbf{x} = (\hat{\mathbf{x}}, x_n)$ .

Recall that a *graph* of a smooth function  $x_n = \varphi(\hat{\mathbf{x}})$  given in an open set  $V \in \mathbb{R}^{n-1}$  is a set of all points of the form  $(\hat{\mathbf{x}}, \varphi(\hat{\mathbf{x}})) \in \mathbb{R}^n$ . It is clear that *any graph is a smooth hypersurface* for which  $U = V \times \mathbb{R} \subset \mathbb{R}^n$  and  $F(\mathbf{x}) = \varphi(\hat{\mathbf{x}}) - x_n$ , since  $\frac{\partial F}{\partial x_n}(\mathbf{x}) = -1$  for any point  $\mathbf{x} \in U$ .  $\square$

The converse is certainly false. For example, the circle  $x^2 + y^2 = 1$  in the plane is not the graph of any function.



A graph of a smooth function

Nevertheless it will be a graph in the neighbourhood of each of its points (the graph of the function  $y = \sqrt{1-x^2}$  in the neighbourhood of say the point  $(0, 1)$ , the graph of the function  $y = -\sqrt{1-x^2}$  in the neighbourhood of the point  $(0, -1)$ , and the graph of the function  $x = \sqrt{1-y^2}$  in the neighbourhood of the point  $(1, 0)$ ; in the last case the role of the coordinate  $x_n$  is played not by the coordinate  $y$  but by the coordinate  $x$ ).

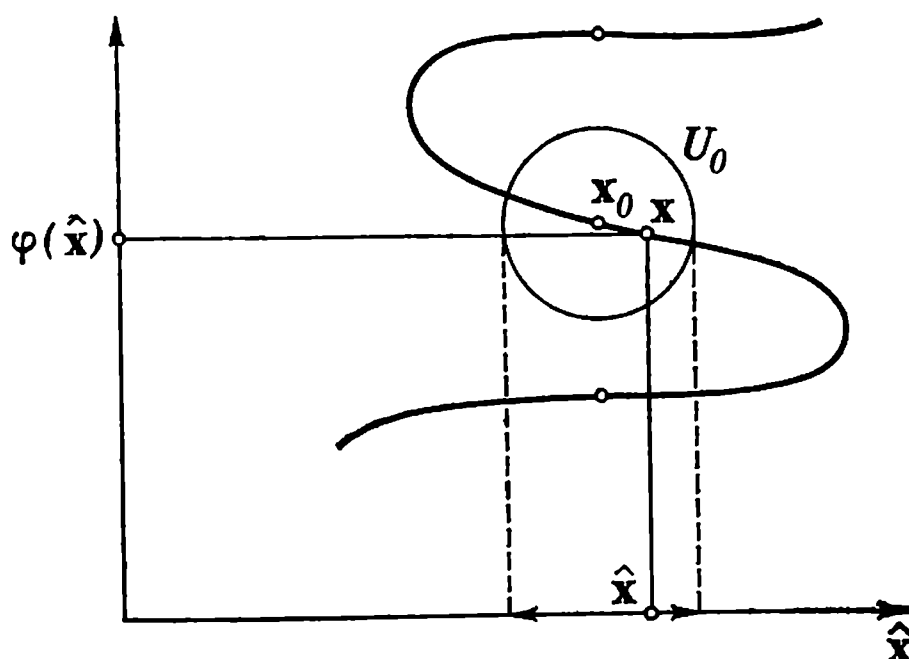
It turns out that a similar statement is true for every hypersurface  $\mathcal{H}$ , i.e. up to an interchange of coordinates *any hypersurface (3) is the graph of some smooth function in the neighbourhood of each of its points*. This statement constitutes the geometrical content of the following theorem known from analysis.

**Implicit function theorem.** Let  $U \subset \mathbb{R}^n$  be an open set,  $\mathbf{x}_0 = (x_1^{(0)}, \dots, x_n^{(0)}) \in U$  be some point in it, and  $F: U \rightarrow \mathbb{R}$  be a smooth function in  $U$  (i.e. from  $\mathcal{F}(U)$ ) such that

$$F(\mathbf{x}_0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial x_n}(\mathbf{x}_0) \neq 0.$$

Then in the space  $\mathbb{R}^n$ , there exists a neighbourhood  $U_0 \subset U$  of the point  $\mathbf{x}_0$  and a function  $x_n = \varphi(\hat{\mathbf{x}})$  defined and smooth in the neighbourhood  $\hat{U}_0 \subset \mathbb{R}^{n-1}$  of a point  $\hat{\mathbf{x}}_0 = (x_1^{(0)}, \dots, x_{n-1}^{(0)})$  such that

- (a)  $\varphi(\hat{\mathbf{x}}_0) = x_n^{(0)}$ ;
- (b)  $(\hat{\mathbf{x}}, \varphi(\hat{\mathbf{x}})) \in U_0$  for any point  $\hat{\mathbf{x}} \in \hat{U}_0$ ;
- (c) if  $\mathbf{x} = (\hat{\mathbf{x}}, x_n) \in U_0$ , then  $F(\mathbf{x}) \neq 0$  if and only if  $x_n = \varphi(\hat{\mathbf{x}})$ .  $\square$



Since the graphs of smooth functions of one and two variables seem to fully correspond with the intuitive notion of smooth curves and surfaces, the implicit function theorem shows that the explication given by Definition 1 of the concept of a hypersurface at any rate is not at variance with intuition. Moreover, the class of smooth hypersurfaces is wide enough to be distinguished.

The restriction to the space  $\mathbb{R}^n$  is of course unessential here: the coordinate isomorphism  $\mathcal{A}^n \rightarrow \mathbb{R}^n$  transfers the concept of a smooth hypersurface to an arbitrary  $n$ -dimensional affine (or Euclidean) space  $\mathcal{A}^n$ . It is clear that the requirement of correctness (of the independence from the choice of coordinate isomorphism) is met here.

The situation is different with the concept of a gradient. For its definition (transferred to a space  $\mathcal{A}^n$ ) to be correct,





If  $\mathbf{k} = (k_1, \dots, k_n)$ , then, according to the indirect differentiation rule,

$$\frac{\partial F}{\partial \mathbf{k}} = k_1 \frac{\partial F}{\partial x_1} + \dots + k_n \frac{\partial F}{\partial x_n},$$

i.e.

$$(5) \quad \frac{\partial F}{\partial \mathbf{k}} = (\mathbf{k}, \text{grad } F).$$

It is usual, however, to consider only the case where  $|\mathbf{k}| = 1$ , i.e. where the vector  $\mathbf{k}$  is a unit vector. In that case the derivative  $\frac{\partial F}{\partial \mathbf{k}}$  is also called a derivative of the function  $F$  with respect to the direction of the vector  $\mathbf{k}$ . In this terminology partial derivatives are none other but derivatives with respect to the direction of coordinate axes.

According to formula (5) the number  $\frac{\partial F}{\partial \mathbf{k}}$  attains maximum (with  $|\mathbf{k}| = 1$ ) when the vector  $\mathbf{k}$  is a unit vector of  $\text{grad } F$ . The vector  $\text{grad } F$  is therefore said to have the direction of the swiftest growth of the function  $F$ .

Note that formula (5), although involving scalar multiplication, does not in fact assume any Euclidean property. Indeed, its right hand side is obviously none other but the value of the gradient  $\text{grad } F$ , regarded as a covector, on the vector  $\mathbf{k}$ . As to the derivative  $\frac{\partial F}{\partial \mathbf{k}}$ , its definition does not assume any Euclidean property at all.

Of course the vector  $\text{grad } F$  in general changes from point to point, i.e. is a vector-valued function in  $U$ . Such functions are called "vector fields". We shall give a general definition of them.

**Definition 2.** Every family  $X$  consisting of  $n$  functions

$$\mathbf{x} \mapsto X_i(\mathbf{x}), \quad i = 1, \dots, n,$$

where  $\mathbf{x} = (x_1, \dots, x_n) \in U$ , is called a *vector field* in  $U$ . A vector field is said to be *smooth* if all functions  $X_i$  are smooth.

Formally a vector field in  $U$  is none other but a smooth mapping  $U \rightarrow \mathbb{R}^n$ .

We have defined a vector field “in an analytical spirit”, i.e. in the space  $\mathbb{R}^n$  with fixed coordinates  $x_1, \dots, x_n$ . In a similar definition for an arbitrary affine (or Euclidean) space  $\mathcal{A}$  one should require that at every point the values of functions  $X_i$  should transform by the vector law when coordinates are changed. We shall not consider such vector fields in  $\mathcal{A}$ , however, since they possess a conceptual defect (as yet hidden from us) and their “proper” definition (with which we shall deal in the lectures of the third semester) is in fact somewhat different.

Yet we venture to write for clearness

$$(6) \quad X = X_1 e_1 + \dots + X_n e_n$$

meaning by  $e_1, \dots, e_n$  a standard basis  $(1, 0, \dots, 0), \dots, (0, \dots, 0, 1)$  of the space  $\mathbb{R}^n$ .

In particular, in that notation

$$\text{grad } F = \frac{\partial F}{\partial x_1} e_1 + \dots + \frac{\partial F}{\partial x_n} e_n.$$

**Definition 3.** A point  $x_0 \in U$  is said to be a *singular point* of a (smooth) vector field  $X$  if  $X_i(x_0) = 0$  for any  $i = 1, \dots, n$ , i.e. if  $X(x_0) = 0$ .

We stress that the field remains smooth at a singular point.

Thus we can say that a *set of points*  $x \in U$  for which  $F(X) = 0$ , where  $F$  is some smooth function, is a *smooth hypersurface* if it does not contain any singular point of the field  $\text{grad } F$ .  $\square$

This set, however, is said to be a hypersurface also when it does contain singular points, provided there are “not too many” of them (otherwise, by virtue of Whitney’s theorem, an arbitrary closed set may result). It is usual to assume that those singular points (called, incidentally, *singular points of the hypersurface*  $F = 0$ ) are isolated or, at worst, fill one or several “surfaces of lower dimension”.

**Example.** The gradient of a quadratic form

$$F(x) = \lambda_1 x_1^2 + \dots + \lambda_n x_n^2, \quad \lambda_1 \neq 0, \dots, \lambda_n \neq 0$$

is expressed by the formula

$$\text{grad } F = (2\lambda_1 x_1, \dots, 2\lambda_n x_n)$$

and has a singular point only at the zero  $(0, \dots, 0)$ . Therefore a *nonsingular second degree hypersurface*

$$\lambda_1 x_1^2 + \dots + \lambda_n x_n^2 = 1$$

(an *ellipsoid* or a *hyperboloid*) has no singular points, i.e. is a smooth hypersurface in the sense of Definition 1.

On the contrary, a second degree cone

$$\lambda_1 x_1^2 + \dots + \lambda_n x_n^2 = 0$$

has a unique singular point, the vertex  $(0, \dots, 0)$ .

A cylinder over a cone

$$\lambda_1 x_1^2 + \dots + \lambda_r x_r^2 = 0, \quad \lambda_1 \neq 0, \dots, \lambda_r \neq 0$$

has an  $n - r$ -dimensional plane  $x_1 = 0, \dots, x_r = 0$  of singular points.

Vector fields can be added:

$$(X + Y)_i(\mathbf{x}) = X_i(\mathbf{x}) + Y_i(\mathbf{x}), \quad i = 1, \dots, n,$$

and multiplied by functions:

$$(fX)_i(\mathbf{x}) = f(\mathbf{x}) X_i(\mathbf{x}), \quad i = 1, \dots, n.$$

An automatic check shows that under these operations the set  $\mathcal{X}(U)$  of all smooth vector fields in  $U$  is a module over the ring  $\mathcal{F}(U)$ .  $\square$

It is appropriate to give one general-algebraic definition here.

Let  $\Lambda$  be an arbitrary ring and  $\mathfrak{M}$  some module over the ring  $\Lambda$ . A family  $m_1, \dots, m_n$  of elements of the module  $\mathfrak{M}$  is said to be its *basis* if for any element  $m \in \mathfrak{M}$  there exist uniquely determined elements  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that

$$m = \lambda_1 m_1 + \dots + \lambda_n m_n.$$

Unlike vector spaces (modules over a field), *not any* module over a ring  $\Lambda$  has a basis. Modules for which there is a basis are called *free*. If all the bases of a free module  $\mathfrak{M}$  consist of the same number  $n$  of elements, the *module*  $\mathfrak{M}$  is said to possess a *rank* and that rank to be equal to  $n$ . In general, there are rings over which there are free modules possessing no rank, but such rings are necessarily noncommutative

(try to prove it!). Therefore, in particular, *any free module possesses a rank over a ring  $F(U)$* .  $\square$

In formula (6) every vector  $e_i$  may be interpreted as a vector field all of whose components are identically zero, except the  $i$ th component which is identically equal to unity. Then the formula will imply that the fields  $e_1, \dots, e_n$  constitute a basis of the module  $\mathcal{X}(U)$ . This proves that for any open set  $U \subset \mathbb{R}^n$  the module  $\mathcal{X}(U)$  of vector fields in  $U$  is a free module of rank  $n$  over the ring  $\mathcal{F}(U)$ .  $\square$

Moreover, the module  $\mathcal{X}(U)$  is obviously, just as the ring  $\mathcal{F}(U)$ , a vector space over the field  $\mathbb{R}$  (of infinite dimension).

The mapping  $F \mapsto \text{grad } F$  of the ring  $F(U)$  into the module  $\mathcal{X}(U)$  carries, as can easily be seen, a sum over into a sum and a product by a number into a product by a number, i.e. is a *linear mapping* (a homomorphism) of the vector space  $\mathcal{F}(U)$  into the vector space  $\mathcal{X}(U)$ . It acts on the product of functions, as follows directly from the formula for differentiating a product, by the formula

$$(7) \quad \text{grad}: (FG) = F \text{ grad } G + G \text{ grad } F.$$

It is obvious that the *kernel of a linear mapping*

$$\text{grad}: F \mapsto \text{grad } F$$

*consists of locally constant functions*, i.e. functions constant on each connected component of a set  $U$ .

The image of a mapping  $\text{grad}$  does not in general coincide with  $\mathcal{X}(U)$ .

**Definition 4.** A vector field of the form  $\text{grad } F$  is called a *gradient*, or *potential*, field. If  $X = \text{grad } F$ , then the function  $F$  is said to be a *potential* of the field  $X$ . The potential (if there is one) is uniquely determined up to a locally constant function.

The vector field (6) is said to be *irrotational* if

$$(8) \quad \frac{\partial X_i}{\partial x_j} = \frac{\partial X_j}{\partial x_i}$$

for any  $i, j = 1, \dots, n$  everywhere in  $U$ . It is easy to see that *every potential field is irrotational*. Indeed, if  $X_i = \frac{\partial F}{\partial x_i}$ , then  $\frac{\partial X_i}{\partial x_j} = \frac{\partial^2 F}{\partial x_i \partial x_j}$  and  $\frac{\partial X_j}{\partial x_i} = \frac{\partial^2 F}{\partial x_j \partial x_i}$ , but, accord-

ing to the familiar property of mixed partial derivatives

$$\frac{\partial^2 F}{\partial x_i \partial x_j} = \frac{\partial^2 F}{\partial x_j \partial x_i}. \quad \square$$

In analysis, instead of the vector field (6) one often prefers to consider the differential expression  $X_1 dx_1 + \dots + \dots + X_n dx_n$ , and then conditions (8) are necessary for that expression to be a *total differential*  $dF$  of some function  $F$ . We shall return to this matter in the third semester's lectures.

Generally speaking, the necessary conditions (8) are insufficient i.e. not any irrotational field is potential. This is so only for the simplest domains  $U$ , such as the interior of a ball or cube. But for arbitrary domains  $U \subset \mathbb{R}^n$  the dimension of the factor space

$$(\text{vector space of irrotational fields}) / (\text{vector space of potential fields})$$

may serve as a measure of their complexity. This remark will also be expounded in the third semester's lectures.

For  $n = 3$  irrotational fields can be described in a more convenient manner. From here (and to the end of the lecture) we assume that  $n = 3$ . By tradition vector fields in  $\mathbb{R}^3$  will be designated  $\mathbf{u}$ ,  $\mathbf{v}$ ,  $\dots$  and so on. The components of a field  $\mathbf{u}$  will be designated (also by tradition)  $P$ ,  $Q$ ,  $R$ , the coordinates  $x_1$ ,  $x_2$ ,  $x_3$  in  $\mathbb{R}^3$  as  $x$ ,  $y$ ,  $z$ , the coordinate unit vectors  $\mathbf{e}_1$ ,  $\mathbf{e}_2$ ,  $\mathbf{e}_3$  as  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  and the vector  $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$  as  $\mathbf{r}$ . As before  $U$  will designate an open set  $U \subset \mathbb{R}^3$  in which all our fields and functions are defined (and smooth).

Note that in this notation]

$$(9) \quad \text{grad } F = \frac{\partial F}{\partial x} \mathbf{i} + \frac{\partial F}{\partial y} \mathbf{j} + \frac{\partial F}{\partial z} \mathbf{k}.$$

**Definition 5.** The *rotation*  $\text{rot } \mathbf{u}$  of a vector field

$$\mathbf{u} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

is a vector field

$$(10) \quad \text{rot } \mathbf{u} = \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) \mathbf{i} + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) \mathbf{j} + \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \mathbf{k}.$$

The symbol  $\text{curl } \mathbf{u}$  was formerly used instead of  $\text{rot } \mathbf{u}$ , but now it has gone out of use\*.

It is clear that the mapping

$$\text{rot}: \mathcal{X}(U) \rightarrow \mathcal{X}(U)$$

is a homomorphism (a linear operator). Its kernel consists exactly of irrotational fields, and the statement that any potential field is irrotational implies that

$$(11) \quad \text{rot grad } F = 0$$

for any function  $F \in \mathcal{F}(U)$ .

**Example 1.** A field of the form

$$\mathbf{u} = f(r) \mathbf{r},$$

where  $r = |\mathbf{r}|$  and  $f$  is an arbitrary smooth (provided  $r > 0$ ) function, is called a *central field*. It is defined and smooth everywhere, except for the point  $(0, 0, 0)$ .

For that field

$$P = f(r) x, \quad Q = f(r) y, \quad R = f(r) z.$$

On the other hand, differentiating the formula  $r = \sqrt{x^2 + y^2 + z^2}$  we immediately have

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Hence

$$\begin{aligned} \frac{\partial P}{\partial x} &= f'(r) \frac{x^2}{r} + f(r), \quad \frac{\partial P}{\partial y} = f'(r) \frac{xy}{r}, \quad \frac{\partial P}{\partial z} = f'(r) \frac{xz}{r}, \\ (12) \quad \frac{\partial Q}{\partial x} &= f'(r) \frac{xy}{r}, \quad \frac{\partial Q}{\partial y} = f'(r) \frac{y^2}{r} + f(r), \quad \frac{\partial Q}{\partial z} = \\ &= f'(r) \frac{yz}{r}, \\ \frac{\partial R}{\partial x} &= f'(r) \frac{xz}{r}, \quad \frac{\partial R}{\partial y} = f'(r) \frac{yz}{r}, \\ \frac{\partial R}{\partial z} &= f'(r) \frac{z^2}{r} + f(r). \end{aligned}$$

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\* The usage in the USSR is meant by the author.—Tr,

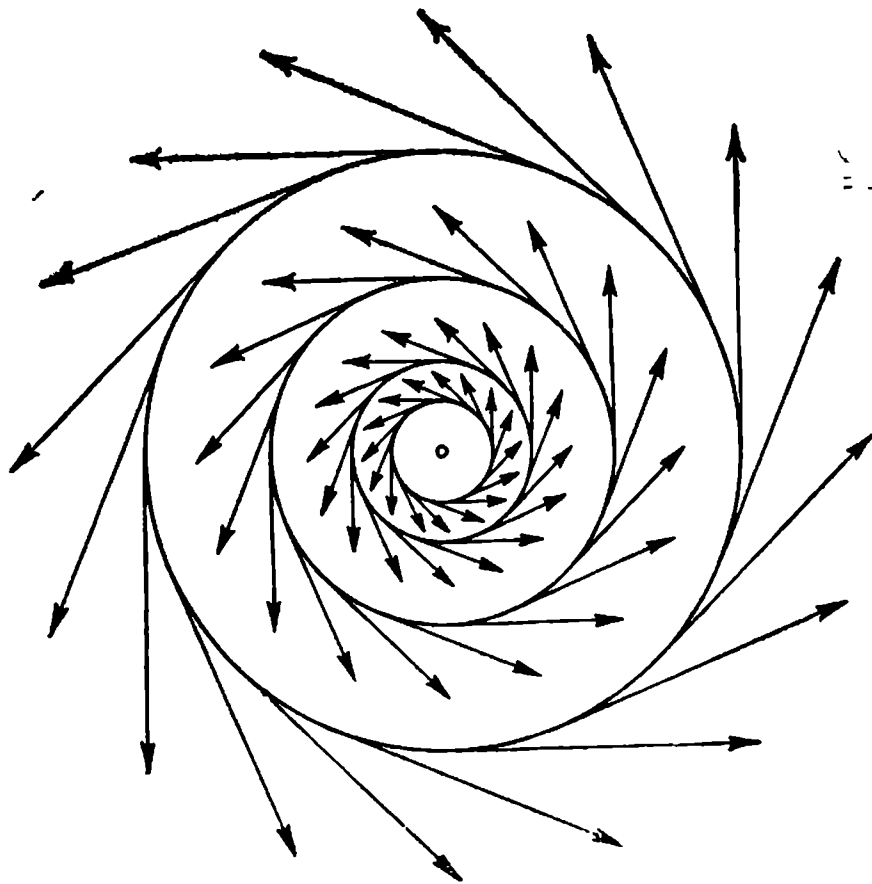
Therefore, in particular,

$$\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}, \quad \frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}, \quad \frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y},$$

i.e.  $\text{rot } \mathbf{u} = 0$ . Thus *every central field is an irrotational field*. Moreover, it turns out that *every central field is potential*. Indeed, by setting

$$F(\mathbf{r}) = \int_1^r r f(r) dr$$

we immediately have  $\mathbf{u} = \text{grad } F$ .  $\square$



*The velocity field of a plane rotation*

If in particular  $f(r) = 1/r^3$  and hence  $|\mathbf{u}| = 1/r^2$  (the gravitational field of a material point), then (up to a constant)  $F(\mathbf{r}) = -1/r$  (the *Newtonian potential*).

**Example 2.** Let

$$P = -y, \quad Q = x, \quad R = 0$$

(the velocity field of a plane rotation). Then

$$\text{rot } \mathbf{u} = 2\mathbf{k},$$



Under multiplication by functions we have for the operator  $\text{rot}$

$$(13) \quad \text{rot} (F\mathbf{u}) = F \text{rot} \mathbf{u} + \text{grad} F \times \mathbf{u}$$

which can be checked by direct computation.

Here by a vector product of two fields we naturally mean a field resulting when we have performed vector multiplication of those fields at every point.

A field which is a rotation, i.e. has the form  $\text{rot} \mathbf{u}$ , is called *solenoidal* (derived from the Greek word *sölën*, tube). If  $\mathbf{v} = \text{rot} \mathbf{u}$ , then the field  $\mathbf{u}$  is called the *vector potential* of the field  $\mathbf{v}$ . It is uniquely determined up to a term which is an irrotational field, i.e. has the form  $\text{grad} F$ , in the simplest domains  $U$ .

**Definition 6.** The *divergence*  $\text{div} \mathbf{u}$  of a vector field

$$\mathbf{u} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$$

is the function

$$(14) \quad \text{div} \mathbf{u} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

The field  $\mathbf{u}$  is said to be a *field without sources* if the function  $\text{div} \mathbf{u}$  is identically zero.

**Example 3.** For a central field  $\mathbf{u} = f(r)\mathbf{r}$  we have (see formulas (12))

$$(15) \quad \text{div} \mathbf{u} = 3f(r) + rf'(r).$$

When  $f(r) = 1/r^3$  this expression is equal to zero. Thus the *force field of a Newtonian potential has no sources*.

An automatic computation shows that

$$\text{div} \text{rot} \mathbf{u} = 0$$

for any field  $\mathbf{u} \in \mathcal{X}(U)$ . Thus *every solenoidal field is a field without sources*.  $\square$

Again the converse is true only for fairly “simple” domains, and again the dimension of the factor space

$$(\text{vector space of fields without sources}) / (\text{vector space of solenoidal fields})$$

can serve as a measure of complexity of a domain  $U$ .

The mapping

$$\operatorname{div}: \mathcal{X}(U) \rightarrow \mathcal{F}(U)$$

is obviously linear and, as is shown by a direct check,

$$(16) \quad \operatorname{div}(Fu) = F \operatorname{div} u + u \operatorname{grad} F$$

for any function  $F \in \mathcal{F}(U)$  and any field  $u \in \mathcal{X}(U)$ .

Thus we have defined three linear mappings:

$$\mathcal{F}(U) \xrightarrow{\operatorname{grad}} \mathcal{X}(U) \xrightarrow{\operatorname{rot}} \mathcal{X}(U) \xrightarrow{\operatorname{div}} \mathcal{F}(U)$$

possessing properties (7), (13) and (16) and such that compositions  $\operatorname{rot} \circ \operatorname{grad}$  and  $\operatorname{div} \circ \operatorname{rot}$  are zero.

The theory of these linear mappings is known as *vector analysis*. It plays an especially important role in the theory of electromagnetism in physics.

Every electromagnetic field (for example, light) is given at each point of a medium by two vectors, an electric vector  $\mathbf{E}$  and a magnetic vector  $\mathbf{H}$ . These vectors depend not only on the point, but also on time  $t$  and are completely defined if we know the electric charge density  $\rho$  and the vector field  $\mathbf{j}$  of current density. Equations relating  $\mathbf{E}$  and  $\mathbf{H}$  to  $\rho$  and  $\mathbf{j}$  have (in the corresponding system of units) the form

$$\begin{aligned} \operatorname{div} \mathbf{E} &= 4\pi\rho, & \operatorname{div} \mathbf{H} &= 0, \\ \operatorname{rot} \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{H}}{\partial t}, & \operatorname{rot} \mathbf{H} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c} \mathbf{j}, \end{aligned}$$

where  $c$  is the velocity of light. These equations called the *Maxwell equations* underlie the entire theory of electromagnetism and, in particular, that of optics and radio engineering. As a matter of fact, vector analysis was first developed as a tool for investigating these equations. However, it has been successfully used say in continuum mechanics as well, and is of course of no small purely mathematical importance.

The most important chapters of vector analysis are connected with the so-called *integral formulas* which we shall discuss in the third semester's lectures. For the time being, however, we shall consider only the simplest formulas of vector analysis that use no integrals.

To derive these formulas, it is appropriate to introduce what is called *Hamilton's symbolic vector field*

$$\nabla = \frac{\partial}{\partial x} \mathbf{i} + \frac{\partial}{\partial y} \mathbf{j} + \frac{\partial}{\partial z} \mathbf{k}.$$

Assuming that a product of say  $\frac{\partial}{\partial x}$  by a function  $P$  is a partial derivative  $\frac{\partial P}{\partial x}$ , we may consider the right hand side of formula (14) defining the function  $\text{div } \mathbf{u}$  as a scalar product of a field  $\nabla$  by a field  $\mathbf{u}$ . Thus

$$\text{div } \mathbf{u} = \nabla \mathbf{u}.$$

Similarly a field  $\text{rot } \mathbf{u}$  may be represented as a vector product

$$\text{rot } \mathbf{u} = \nabla \times \mathbf{u},$$

which, incidentally, allows us to write for  $\text{rot } \mathbf{u}$  a beautiful determinantal expression:

$$\text{rot } \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

Finally, by allowing the numerical factor to be written at the right of a vector a field  $\text{grad } F$  can also be represented as a product of  $\nabla$  by  $F$ :

$$\text{grad } F = \nabla F.$$

Now let  $\alpha$  and  $\mathbf{b}$  be either functions or vector fields. Then they can be multiplied together in many different ways (for example, if  $\alpha$  and  $\mathbf{b}$  are vector fields, by performing scalar or vector multiplication). Let  $*$  and  $\cdot$  be two multiplications such that the expression  ${}^1\nabla_1^{*2}(\alpha_2^*\mathbf{b})$  is well-defined.

The familiar rule of differentiating a product may be formulated as follows: we differentiate the product twice, differentiating only one factor at a time and then adding

both results together. It is fairly clear that the same rule also applies to an action by the operator  $\nabla$ . Therefore

$$(17) \quad \nabla_1^* (\alpha_2^* \flat) = \nabla_1^* (\alpha_2^* \overset{\downarrow}{\flat}) + \nabla_1^* (\overset{\downarrow}{\alpha_2^*} \flat),$$

where the vertical arrow marks the factor acted upon by the operator  $\nabla$ .

Let, for example,  $\alpha$  and  $\flat$  be functions  $F$  and  $G$  (and hence let  $_2^*$  be the multiplication of functions and  $_1^*$  the multiplication of a vector field by a function). Then

$$\nabla (FG) = \nabla (F \overset{\downarrow}{G}) + \nabla (\overset{\downarrow}{F} G).$$

But it is clear that  $\nabla (F \overset{\downarrow}{G}) = F (\nabla G)$  and similarly  $\nabla (\overset{\downarrow}{F} G) = G (\nabla F)$ . Hence

$$\nabla (FG) = F (\nabla G) + G (\nabla F).$$

It is the familiar formula (7).

If  $\alpha$  is a function  $F$  and  $\flat$  is a field  $\mathbf{u}$ , then formula (17) yields

$$\nabla (F\mathbf{u}) = \nabla (F \overset{\downarrow}{\mathbf{u}}) + \nabla (\overset{\downarrow}{F} \mathbf{u})$$

and

$$\nabla \times (F\mathbf{u}) = \nabla \times (F \overset{\downarrow}{\mathbf{u}}) + \nabla \times (\overset{\downarrow}{F} \mathbf{u}).$$

In the first formula  $\nabla (F \overset{\downarrow}{\mathbf{u}}) = F (\nabla \mathbf{u})$  and  $\nabla (\overset{\downarrow}{F} \mathbf{u}) = (\nabla F) \mathbf{u}$ , so that

$$\nabla (F\mathbf{u}) = F (\nabla \mathbf{u}) + (\nabla F) \mathbf{u}.$$

It is formula (16).

Similarly, in the second formula  $\nabla \times (F \overset{\downarrow}{\mathbf{u}}) = F (\nabla \times \mathbf{u})$  and  $\nabla \times (\overset{\downarrow}{F} \mathbf{u}) = (\nabla F) \times \mathbf{u}$  and hence

$$\nabla \times (F\mathbf{u}) = F (\nabla \times \mathbf{u}) + \nabla F \times \mathbf{u}.$$

It is formula (13).

Finally, if  $a$  and  $b$  are fields  $u$  and  $v$ , then three new formulas result:

$$(17a) \quad \nabla(uv) = \nabla(u \overset{\downarrow}{v}) + \nabla(\overset{\downarrow}{u}v),$$

$$(17b) \quad \nabla(u \times v) = \nabla(u \times \overset{\downarrow}{v}) + \nabla(\overset{\downarrow}{u} \times v),$$

$$(17c) \quad \nabla \times (u \times v) = \nabla \times (u \times \overset{\downarrow}{v}) + \nabla \times (\overset{\downarrow}{u} \times v).$$

Formula (17b) is the easiest to decipher. Indeed, using the properties of a triple product we immediately get

$$\nabla(u \times \overset{\downarrow}{v}) = \nabla u \overset{\downarrow}{v} = -u \nabla \overset{\downarrow}{v} = -u \nabla v = -u(\nabla \times v)$$

and

$$\nabla(\overset{\downarrow}{u} \times v) = \nabla \overset{\downarrow}{u} v = v \nabla \overset{\downarrow}{u} = v \nabla u = v(\nabla \times u).$$

Hence formula (17b) is equivalent to the formula

$$(18) \quad \operatorname{div}(u \times v) = (\operatorname{rot} u) v - u \operatorname{rot} v.$$

Of course that “hence” is highly arbitrary, since we have in no way substantiated the validity of applying the properties of a triple product to products containing a symbolic field  $\nabla$ . Such a substantiation would lead us too far away besides requiring supplementing with a more detailed justification of the original formula (17) which strictly speaking was assumed above virtually without proof. We are therefore justified in regarding all the foregoing as nothing but mnemonic or at best heuristic considerations combining in a single formula, (17), the hitherto entirely unrelated formulas (7), (16), (13) and (18). As to the formal proof of these last formulas (and, in particular, of the new formula (18)), nothing remains but to check each of them independently by direct calculation.

The possibilities of formula (17) are not restricted to the four formulas listed: we have not yet deciphered the two symbolic formulas (17a) and (17c). We shall need the following lemma to transform them.

*Lemma. For any three vectors  $a$ ,  $b$ ,  $c$  the formula*

$$(19) \quad c \times (a \times b) = (cb) a - (ac) b$$

*is valid.*

**Proof.** Choose an arthonormal basis  $\mathbf{i}, \mathbf{j}, \mathbf{k}$  such that the vector  $\mathbf{a}$  is collinear with the vector  $\mathbf{i}$  and the vector  $\mathbf{b}$  is coplanar with the vectors  $\mathbf{i}$  and  $\mathbf{j}$ . Then

$$\mathbf{a} = a_1 \mathbf{i},$$

$$\mathbf{b} = b_1 \mathbf{i} + b_2 \mathbf{j},$$

$$\mathbf{c} = c_1 \mathbf{i} + c_2 \mathbf{j} + c_3 \mathbf{k},$$

and therefore

$$\mathbf{a} \times \mathbf{b} = (a_1 b_2) \mathbf{k},$$

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (a_1 b_2 c_2) \mathbf{i} - (a_1 b_2 c_1) \mathbf{j}.$$

On the other hand,

$$\mathbf{cb} = c_1 b_1 + c_2 b_2, \quad \mathbf{ac} = a_1 c_1$$

and therefore

$$\begin{aligned} (\mathbf{cb}) \mathbf{a} - (\mathbf{ac}) \mathbf{b} &= (c_1 b_1 + c_2 b_2) a_1 \mathbf{i} - a_1 c_1 (b_2 \mathbf{i} + b_2 \mathbf{j}) = \\ &= (c_2 b_2 a_1) \mathbf{i} - (a_1 c_1 b_2) \mathbf{j}. \end{aligned}$$

Hence  $\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{cb}) \mathbf{a} - (\mathbf{ac}) \mathbf{b}$ .  $\square$

We shall apply the lemma to the case where one of the factors is a vector field  $\nabla$ , i.e. again merely for purely heuristic-mnemonic purposes. Besides, to obtain the right formulas we have to give another value to the expression  $\mathbf{a}\nabla$ , where  $\mathbf{a} = A\mathbf{i} + B\mathbf{j} + C\mathbf{k}$  is some vector field, different from the one,  $\nabla\mathbf{a} = \text{div } \mathbf{a}$ , suggesting itself, i.e. to give up the commutativity of scalar multiplication in the case of a symbolic vector field  $\nabla$ .

Namely, we shall consider the expression  $\mathbf{a}\nabla$  to be an operator acting on a vector field  $\mathbf{u} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$  by the formula

$$(\mathbf{a}\nabla) \mathbf{u} = A \frac{\partial P}{\partial x} + B \frac{\partial Q}{\partial y} + C \frac{\partial R}{\partial z}.$$

Adopting this convention we get, in view of (19),

$$\nabla \times (\mathbf{u} \times \overset{\downarrow}{\mathbf{v}}) = (\nabla \mathbf{v}) \mathbf{u} - (\mathbf{u}\nabla) \mathbf{v} = (\text{div } \mathbf{v}) \mathbf{u} - (\mathbf{u}\nabla) \mathbf{v},$$

$$\nabla \times (\overset{\downarrow}{\mathbf{u}} \times \mathbf{v}) = (\mathbf{v}\nabla) \mathbf{u} - (\nabla \mathbf{u}) \mathbf{v} = (\mathbf{v}\nabla) \mathbf{u} - (\text{div } \mathbf{u}) \mathbf{v},$$

and thus formula (17c) yields

$$(20) \quad \text{rot} (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \nabla) \mathbf{u} - (\mathbf{u} \nabla) \mathbf{v} + (\text{div } \mathbf{v}) \mathbf{u} - (\text{div } \mathbf{u}) \mathbf{v}.$$

To transform in a similar manner formula (17a) we apply formula (19) after rewriting it in the following form:

$$\mathbf{c} \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a} (\mathbf{c} \mathbf{b}) - (\mathbf{c} \mathbf{a}) \mathbf{b}.$$

Then we get

$$\mathbf{u} \times (\nabla \times \mathbf{v}) = \nabla (\mathbf{u} \mathbf{v}) - (\mathbf{u} \nabla) \mathbf{v}$$

and

$$\mathbf{v} \times (\nabla \times \mathbf{u}) = \nabla (\mathbf{u} \mathbf{v}) - (\mathbf{v} \nabla) \mathbf{u}$$

and therefore

$$\nabla (\mathbf{u} \mathbf{v}) + \nabla (\mathbf{u} \mathbf{v}) = \mathbf{u} \times (\nabla \times \mathbf{v}) + \mathbf{v} \times (\nabla \times \mathbf{u}) + (\mathbf{v} \nabla) \mathbf{u} + (\mathbf{u} \nabla) \mathbf{v}.$$

Thus formula (17a) yields the formula

$$(21) \quad \text{grad} (\mathbf{u} \mathbf{v}) = \mathbf{u} \times \text{rot } \mathbf{v} + \mathbf{v} \times \text{rot } \mathbf{u} + (\mathbf{v} \nabla) \mathbf{u} + (\mathbf{u} \nabla) \mathbf{v}.$$

Of course, formal proofs of formulas (20) and (21) must as before consist in direct calculations.

Interesting relations hold for compositions of operators grad, rot and div as well.

As we already know

$$\text{rot} \circ \text{grad} = 0,$$

$$\text{div} \circ \text{rot} = 0.$$

Note that an operator  $\nabla$  reduces these formulas to the assertion that a vector product of two equal vectors is zero:

$$\text{rot} (\text{grad } F) = \nabla \times (\nabla F) = (\nabla \times \nabla) F = 0$$

and

$$\text{div} (\text{rot } \mathbf{u}) = \nabla (\nabla \times \mathbf{u}) = (\nabla \times \nabla) \mathbf{u} = 0.$$

Of special interest is the operator

$$\Delta = \text{div} \circ \text{grad}$$

which may be regarded as the scalar square  $\nabla^2$  of a Hamiltonian operator  $\nabla$ . This operator is called a *Laplacian operator*. It is an operator from  $\mathcal{F}(U)$  into  $\mathcal{F}(U)$  and acts by

$$\Delta F = \frac{\partial^2 F}{\partial x^2} + \frac{\partial^2 F}{\partial y^2} + \frac{\partial^2 F}{\partial z^2}.$$

It is used to write the most important equations of mathematical physics, to which a separate course is devoted in the curricula of universities.

A function  $F$  is said to be *harmonic* if  $\Delta F = 0$ . An example of a harmonic function is the Newtonian potential  $F = -1/r$  (see above). As will be shown in the course in mathematical-physics equations, *any harmonic function is the potential of the gravitational field of some mass*. This alone shows the important role played by harmonic functions in physics (and hence also in mathematics).

The operator  $\Delta$  can be applied to vector fields as well by acting with it on every individual component: if  $\mathbf{u} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ , then

$$\Delta \mathbf{u} = (\Delta P)\mathbf{i} + (\Delta Q)\mathbf{j} + (\Delta R)\mathbf{k}.$$

Then the following formula holds

$$\text{rot} \circ \text{rot} = \text{grad} \circ \text{div} - \Delta.$$

Indeed, according to the lemma

$$\text{rot}(\text{rot } \mathbf{u}) = \nabla \times (\nabla \times \mathbf{u}) = \nabla(\nabla \cdot \mathbf{u}) - (\nabla \nabla) \mathbf{u}. \quad \square$$

It is possible to set up other differential expressions as well. For example, for any two functions  $F$  and  $G$  the scalar product of their gradients is defined:

$$\Delta(F, G) = \text{grad } F \cdot \text{grad } G = \frac{\partial F}{\partial x} \frac{\partial G}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial G}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial G}{\partial z}.$$



It is called *Beltrami's mixed differential parameter* of the functions  $F$  and  $G$ . In particular, when  $F = G$  we obtain the scalar square of a gradient:

$$\Delta_1 F = (\text{grad } F)^2 = \left( \frac{\partial F}{\partial x} \right)^2 + \left( \frac{\partial F}{\partial y} \right)^2 + \left( \frac{\partial F}{\partial z} \right)^2.$$

It is called *Beltrami's first differential parameter* of the function  $F$ .

The triple product of the gradients of three functions is called *Darboux's differential parameter*. This term, however, is almost completely out of use, since the triple product is nothing but the *Jacobian* of a transformation defined by three given functions.

# Lecture 23

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*Continuous, smooth, and regular curves • Equivalent curves • Regular curves in the plane and graphs of functions • The tangential hyperplane of a hypersurface • The length of a curve • Curves in the plane • Curves in three-dimensional space*

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Explicating the notion of a curve as the trajectory of a point we obtain the following definition.

**Definition 1.** A continuous curve in an  $n$ -dimensional Euclidean (or affine) space  $\mathcal{E}$  is a continuous mapping

$$(1) \quad \mathbf{x}: t \mapsto \mathbf{x}(t)$$

of some closed interval  $[a, b]$ ,  $a < b$ , of the axis  $t$  in the space  $\mathcal{E}$  (meaning that points of the space  $\mathcal{E}$  are characterized by their radius vectors with respect to a fixed point  $O$ ).

It makes sense to speak of the continuity of mappings of the form (1), since the Euclidean space  $\mathcal{E}$  is a metric space. It can easily be shown (do it!) that the continuity of mapping (1) is equivalent to the continuity of  $n$  numerical functions

$$(2) \quad x_i: t \mapsto x_i(t), \quad i = 1, \dots, n,$$

where  $x_1(t), \dots, x_n(t)$  are the coordinates of a vector  $\mathbf{x}(t)$  in an arbitrary basis. Since the basis is *a priori* in no way connected with any metric (is not orthonormal), we see that mapping (1) continuous in one metric is so in any other. This means that the continuity property of mapping (1) does not depend on any metric and is therefore an affine property. In other words, it makes sense to speak of continuous mappings of the form (1) also when  $\mathcal{E}$  is an affine

space (and hence is not a metric space). Cf. Definition 1 of Lecture 12 in [1].

The reader must already know Definition 1 (as applied to the space  $\mathbb{R}^n$ ) from the course in analysis.

We stress that according to this definition a continuous curve is a mapping, not a set of points. Nevertheless, one uses such terminology referring to curves as if they were sets. Thus curve (1) is said *to pass through a point*  $\mathbf{x}_0$  if there exists (generally speaking, more than one) value  $t_0$  of the parameter  $t$  such that  $\mathbf{x}(t_0) = \mathbf{x}_0$ . The point  $\mathbf{x}(a)$  is called the *initial point* of curve (1) and the point  $\mathbf{x}(b)$  is its *terminal point*. Also curve (1) is said *to connect* the point  $\mathbf{x}(a)$  to the point  $\mathbf{x}(b)$  and so on.

The set of all points of curve (1), i.e. the image of the interval  $[a, b]$  under mapping (1), is sometimes called the *support* of curve (1).

Definition 1 was proposed as early as the last century by the French mathematician Jordan who was certain (and this certainty of his was shared by all mathematicians) that it reflects fairly well the intuitive notion of a curve. But soon all mathematical world was astounded at the news that the Italian mathematician Peano had constructed a continuous curve that passes (several times, in fact) through each (!) point of a square. It became clear that the continuity condition alone is not enough and that some other, additional conditions are necessary.

In our previous lecture we introduced the concept of a function  $F$  smooth on some open set  $U \subset \mathbb{R}^n$ . Now consider an arbitrary set  $C \subset \mathbb{R}^n$  and some function  $f$  given on  $C$ . We say that the function  $f$  is a *function smooth on  $C$*  if there exists an open set  $U \subset \mathbb{R}^n$  and a function  $F$  smooth on  $U$  such that  $C \subset U$  and

$$f = F|_C.$$

In particular, a function  $t \mapsto \mathbf{x}(t)$  given on the interval  $[a, b]$  will be said to be *smooth* if on some open interval containing the closed interval  $[a, b]$  there exists a smooth function coinciding on  $[a, b]$  with the function  $\mathbf{x}(t)$ .

**Definition 2.** Mapping (1) is said to be a *smooth curve* in  $\mathcal{E}$  if the coordinate functions (2) are functions smooth on  $[a, b]$ .

It is obvious that this definition is correct (is independent of the choice of coordinate system). ▮

For any smooth curve (1) and any  $t \in [a, b]$ , there exists a limit

$$(3) \quad \mathbf{x}'(t) = \lim_{\Delta t \rightarrow 0} \frac{\mathbf{x}(t + \Delta t) - \mathbf{x}(t)}{\Delta t}.$$

This limit is called the *tangent vector* of (or to) curve (1) at the point  $t$  (or at the point  $\mathbf{x}(t)$ ). Its coordinates are obviously the derivatives

$$(4) \quad x_1'(t), \dots, x_n'(t)$$

of coordinates (2) of the vector  $\mathbf{x}(t)$ . Vector (3) is also designated by the symbol  $\frac{d\mathbf{x}(t)}{dt}$ .

This construction may obviously be iterated any number of times to yield vectors  $\mathbf{x}''(t)$ ,  $\mathbf{x}'''(t)$ , etc. whose coordinates are the corresponding derivatives of the coordinate functions (2).

It is easy to see that if for some continuous curve (1) limit (3) exists, then so do derivatives (4). Thus the smoothness condition of curve (1) is the existence condition of any derivatives  $\mathbf{x}'(t)$ ,  $\mathbf{x}''(t)$ , ... (we need). This shows once again that the smoothness condition is independent of the choice of coordinate system.

On the whole smooth curves (more precisely, their supports) already correspond to the intuitive idea of a curve. At any rate a smooth curve, as we shall show in the third semester's lectures, cannot pass through all the points of a square (and what is more, the set of all of its points, i.e. its support, is what is called a "set of measure zero"). It may be, however, of not a "smooth" character at all points and possess "cusps", just as, say, the curve  $x = t^2$ ,  $y = t^3$  in the plane does. To avoid such pathologies we introduce the following definition:

**Definition 3.** A smooth curve (1) is said to be *regular* if  $\mathbf{x}'(t) \neq 0$  for all  $t \in [a, b]$ .

Now a regular curve fully corresponds to the intuitive idea of a "smooth" curve. Before discussing this matter, however, we must consider yet another important question.

From an intuitive, geometrically apparent point of view, the main drawback of Definitions 1 to 3 is that the "curves" they introduce are not sets. On the other hand, the definition of a curve as simply an image of the closed interval  $[a, b]$  under its continuous (smooth or regular) mapping into a space  $\mathcal{E}$  turns out, for many reasons, to be quite unsatisfactory. The following definition is usually introduced to approach at least partly the intuitive-geometrical notion of a curve and to obtain at the same time its efficient explication.

**Definition 4.** Two curves

$$\mathbf{x}: t \mapsto \mathbf{x}(t), \quad \mathbf{x}_1: t_1 \mapsto \mathbf{x}_1(t_1).$$

where  $a \leq t \leq b$  and  $a_1 \leq t_1 \leq b_1$  respectively are said to be *equivalent* if there exists a function

$$(5) \quad \varphi: t \mapsto \varphi(t)$$

such that  $\varphi(a) = a_1$ ,  $\varphi(b) = b_1$  and  $\mathbf{x}(t) = \mathbf{x}_1(\varphi(t))$  for all  $t \in [a, b]$ . Function (5) is said to effect a *change* of parameter  $t$ .

It is clear that equivalent curves have the same supports.

Classes of equivalent curves are called *non-parametric curves*. Many authors (mainly of a more traditional slant) call them simply curves, referring to curves in the sense of Definitions 1 to 3 as *parametric curves* or *paths*. Intuitively transition to an equivalent curve means that without changing the trajectory of a point we change the velocity with which it moves along the trajectory. It is clear that this change of velocity cannot be arbitrary. If for example we are considering continuous curves, in principle it is necessary to require that the function  $\varphi$  should effect a homeomorphic (one-to-one and bicontinuous) mapping of the interval  $[a, b]$  onto an interval  $[a_1, b_1]$ , i.e. that it should be a continuous and strictly monotonic function (then the inverse function exists and is also continuous). Otherwise the relation between curves introduced by Definition 4 is not, generally speaking, an equivalence relation on the set of all curves and will not therefore allow introduction of classes of equivalent curves. It is possible, however, to admit functions (5) that are not strictly monotonic and hence discontinuous inverse functions, provided the curve  $t \mapsto$

$\mapsto \mathbf{x}_1(\varphi(t))$  for the discontinuous function  $\varphi$  remains continuous. This means that the point is allowed to stop for a time in moving along the trajectory, and conversely if the point remained fixed, it is allowed to pass the place without stopping in the equivalent motion. Moreover, it is possible, by slightly complicating Definition 4, to admit any nonmonotonic functions (5) too (thus allowing the point to retrace its trajectory). It is usual to discuss all these questions in detail in the course in analysis. But we shall restrict ourselves, in accordance with our general purpose, to *regular changes of parameter*, i.e. to such functions (5) that are, first, smooth and, second, possess the property that

$$\varphi'(t) > 0 \text{ for any } t \in [a, b].$$

This will ensure that the regularity properties are preserved under changes of parameter.

One should not exaggerate the significance of the concept of a nonparametric curve, since, first, it is one order ("an extra equivalence") more complex than the concept of a parametric curve and, second, even in spite of this it does not fully correspond to the intuitive idea of a curve as a set of points (curves may have the same support but fail to be equivalent). At the beginning of this century, of the two concepts of a curve that of a nonparametric curve was considered to be the basic one, as supposedly more apparent geometrically. In recent years, however, parametric curves have more and more often come to the fore not only because they are simpler conceptually, but chiefly because it is these curves that tend to occur in real mathematical constructions. In particular, this explains why the simple word "curves" formerly applied to nonparametric curves is now used more and more often to refer to parametric curves.

A role of no small importance is played of course also by the fact that many natural and convenient concepts and constructions are not preserved under equivalence and cannot therefore be defined for nonparametric curves. The situation is such for example with the concept of a tangent vector which is multiplied by  $\varphi'(t)$  when passing to the equivalent curve. Therefore even ardent advocates of the priority of nonparametric curves pass in practice to para-

metric curves, adducing the “naturalness” (see below) of the concept they are introducing, to excuse their fall.

For these reasons the main subject of our study will be parametric curves and we shall pass to equivalent curves only sporadically and without attaching any significance to this.

Now we are in a position to discuss the question of the extent to which the concept of a regular curve corresponds to the intuitive notion of a curve. For simplicity we shall restrict ourselves to the case of a plane. As always coordinates in the plane will be denoted by  $x$  and  $y$ .

The graph of an arbitrary smooth function  $y = y(x)$  is the support of the regular curve

$$x = t, \quad y = y(t)$$

which we shall also call, loosely but quite naturally, the graph of the function  $y(x)$ .

What curves in the plane satisfy our intuitive idea of a “smooth curve”? It appears to be possible to require that the following conditions should be fulfilled:

(a) the graph of any smooth function (with coordinate axes arbitrarily arranged) is a “smooth curve”;

(b) a curve (regularly) equivalent to a “smooth curve” is a “smooth curve”;

(c) a curve is a “smooth curve” if and only if it is a “smooth curve” locally, i.e. in the neighbourhood of any of its points.

The smallest class of curves that satisfies these conditions consists of curves *locally equivalent* (i.e. equivalent in the neighbourhood of every point) to the graphs of smooth functions (changing from point to point). It is clear that all such curves are regular. It turns out (just this justifies from the intuitive point of view the distinguishing of the class of regular curves) that the converse is also true: *any regular curve in the plane is locally equivalent to the graph of a smooth function.*

Indeed, if the curve

$$(6) \quad x = x(t), \quad y = y(t), \quad a \leq t \leq b,$$

is regular, then for any point  $t_0 \in [a, b]$  either  $x'(t_0) \neq 0$  or  $y'(t_0) \neq 0$ . Let for definiteness  $x'(t_0) \neq 0$ . Then by the

implicit function theorem (applied to the function  $F(x, t) = x - x(t)$ ) the function  $t \mapsto x(t)$  is *locally invertible*, i.e. there exists the neighbourhood  $U_0$  of a point  $t_0$  and the neighbourhood  $V_0$  of a point  $x_0 = x(t_0)$  such that the function  $t \mapsto x(t)$  gives a bijective mapping  $U_0 \mapsto V_0$ , the inverse function  $x \mapsto t(x)$  being smooth. Moreover,  $t'(x) \neq 0$  for all  $x \in V_0$  and hence if  $t'(x) > 0$  in  $V_0$ , then the function  $x \mapsto t(x)$  will effect a regular change of parameter for curve (6) in the neighbourhood  $U_0$ . That change converts curve (6) (in the neighbourhood  $U_0$ ) into an equivalent curve which is (in  $V_0$ ) the graph of a smooth function  $y = y(t(x))$ . But if  $t'(x) < 0$  in  $V_0$ , then it is necessary to take  $-x$  rather than  $x$  as the new parameter (i.e. to change the sense of the abscissa axis) and if  $x'(t_0) = 0$ , then the new parameter will be  $y$  (or  $-y$ ).  $\square$

Note that in this statement "locality" is understood "relative to a parameter", i.e. the restriction of the curve to some neighbourhood of a point  $t_0 \in [a, b]$  is considered. For the neighbourhood of a point  $(x(t_0), y(t_0))$  in the plane a similar statement is even meaningless.

**Example.** The curve

$$x = \frac{3t(1-t)^2}{3t^2-3t+1}, \quad y = \frac{3t^2(1-t)}{3t^2-3t+1}$$

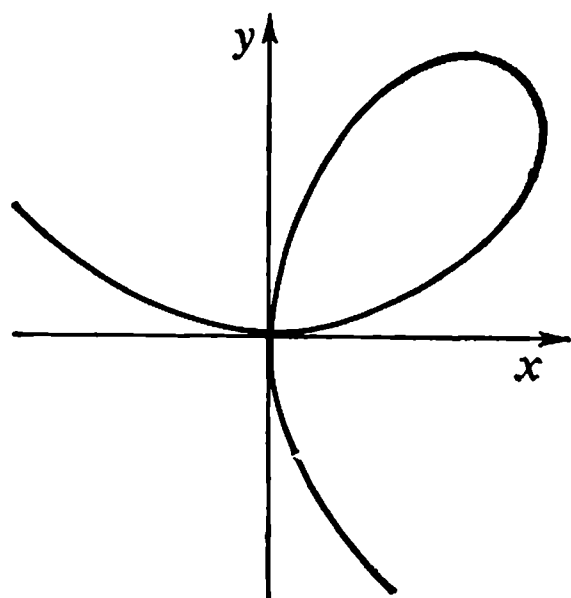
called a *folium of Descartes* passes through the point  $(0, 0)$  twice, at  $t = 0$  and  $t = 1$ . It is equivalent to the graph of some function  $y = y(x)$  in the neighbourhood of the point  $t = 0$  and to the graph of a function  $x = x(y)$  in the neighbourhood of the point  $t = 1$ . But in the neighbourhood of the point  $(0, 0)$  in the plane the curve (or more exactly its support) is a union of these two graphs.

For a folium of Descartes the point  $(0, 0)$  is what is called a *point of self-intersection*. The graphs into which the folium of Descartes breaks in the neighbourhood of the point  $(0, 0)$  are called its *branches*. We shall not dwell on phenomena of this kind since in what follows we confine ourselves to a local study of curves on sufficiently small intervals of the axis  $t$  (i.e., consequently, when they are equivalent to graphs) and merely remark that it is because of the presence of self-intersections that regular curves (or more exactly their supports) will not be regular hypersurfaces in the plane



in the sense of Definition 1 in the preceding lecture. However, we cannot all the same state as yet, of course (outside the limits of local consideration) that the support of every curve in the plane that has no intersections is a regular hypersurface and that, conversely, any regular hypersurface in the plane is the support of a regular curve (automatically without a self-intersection). In the third semester's lectures we shall investigate such questions in their natural generality

and therefore leave them undiscussed for the time being.



*A folium of Descartes*

In the spirit of all the other terminology relating to curves we shall say that curve (1) *lies* (or *is*) on the hypersurface

$$(7) \quad F(\mathbf{x}) = 0$$

of a space  $\mathcal{E}$  if  $F(\mathbf{x}(t)) = 0$  for any  $t \in [a, b]$ , i.e. if hypersurface (7) contains the support of that curve.

**Definition 5.** A vector  $\mathbf{a}$  is said to be the *tangent vector* of (or to) hypersurface (7) at its point  $\mathbf{x}_0$  if on the hypersurface (7) a curve  $t \mapsto \mathbf{x}(t)$  passing through the point  $\mathbf{x}_0$ , with  $t = t_0$ , lies such that  $\mathbf{a}$  is the tangent vector of that curve at the point  $t_0$ , i.e. if  $\mathbf{a} = \mathbf{x}'(t_0)$ .

Let  $\mathcal{V}$  be a vector space associated with the point space  $\mathcal{E}$  (Euclidean, for definiteness) and let  $\mathcal{H}_{\mathbf{x}_0}$  be the set of all vectors tangent to hypersurface (7) (assumed to be regular) at its point  $\mathbf{x}_0$ .

**Proposition 1.** The set  $\mathcal{H}_{\mathbf{x}_0}$  is an  $n - 1$ -dimensional subspace of a space  $\mathcal{V}$  consisting of all the vectors orthogonal to the vector  $\text{grad } F(\mathbf{x}_0)$ :

$$\mathcal{H}_{\mathbf{x}_0} = \{\mathbf{a} \in \mathcal{V}; \mathbf{a} \cdot \text{grad } F(\mathbf{x}_0) = 0\}.$$

**Proof.** If  $\mathbf{a} \in \mathcal{H}_{\mathbf{x}_0}$ , then there exists a curve  $t \mapsto \mathbf{x}(t)$ ,  $a \leq t \leq b$ , in  $\mathcal{V}$  such that

$$(8) \quad F(\mathbf{x}(t)) = 0 \text{ for all } t \in [a, b]$$

and

$$(9) \quad \mathbf{x}_0 = \mathbf{x}(t_0), \quad \mathbf{a} = \mathbf{x}'(t_0).$$

But the formula, known from analysis, for a derivative of the composite function

$$F(\mathbf{x}(t)) = F(x_1(t), \dots, x_n(t))$$

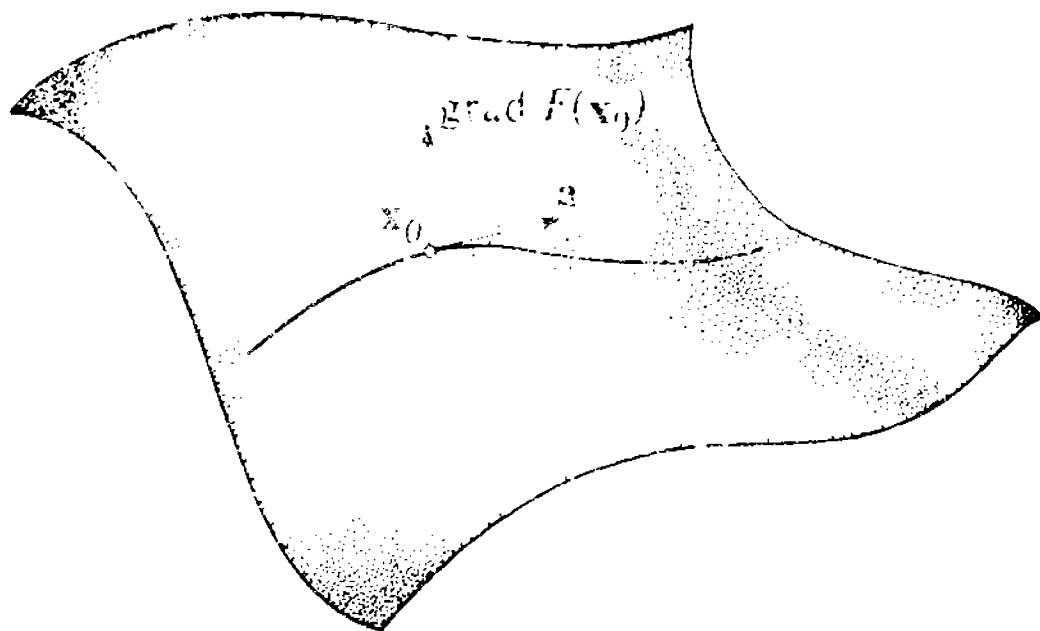
may be written in the form

$$\frac{dF(\mathbf{x}(t))}{dt} = \mathbf{x}'(t) \operatorname{grad} F(\mathbf{x}(t))$$

(we naturally assume the coordinates  $x_1, \dots, x_n$  rectangular). Differentiating relations (8) and putting  $t = t_0$  we therefore get, by virtue of (9),

$$(10) \quad \mathbf{a} \operatorname{grad} F(\mathbf{x}_0) = 0.$$

Conversely, let relation (10) hold. Without loss of generality we may choose a coordinate system so that the vector



*A tangent vector*

$\operatorname{grad} F(\mathbf{x}_0)$  is parallel to the axis  $Ox_n$ . Then the following relations will hold

$$(11) \quad \frac{\partial F}{\partial x_1}(\mathbf{x}_0) = 0, \dots, \frac{\partial F}{\partial x_{n-1}}(\mathbf{x}_0) = 0, \quad \frac{\partial F}{\partial x_n}(\mathbf{x}_0) \neq 0$$

and condition (10) will take the form  $a_n = 0$ . Since  $\frac{\partial F}{\partial x_n}(\mathbf{x}_0) \neq 0$ , in the neighbourhood of the point  $\mathbf{x}_0$  hypersurface (7) is the graph of some smooth function

$$x_n = \varphi(\hat{\mathbf{x}}).$$

This means that

$$x_n^{(0)} = \varphi(x_1^{(0)}, \dots, x_{n-1}^{(0)}).$$

where  $x_1^{(0)}, \dots, x_{n-1}^{(0)}$  are the coordinates of the point  $\mathbf{x}_0$  and

$$F(\hat{\mathbf{x}}, \varphi(\hat{\mathbf{x}})) = 0$$

for all the points  $\hat{\mathbf{x}} = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$  belonging to some neighbourhood  $\hat{U}_0$  of a point  $\hat{\mathbf{x}}_0 \in \mathbb{R}^{n-1}$ . Differentiating the last identity with respect to  $x_1, \dots, x_{n-1}$  and putting  $\hat{\mathbf{x}} = \hat{\mathbf{x}}_0$  we get by virtue of (11)

$$\frac{\partial \varphi}{\partial x_1}(\hat{\mathbf{x}}_0) = 0, \dots, \frac{\partial \varphi}{\partial x_{n-1}}(\hat{\mathbf{x}}_0) = 0.$$

Now let  $\delta > 0$  be a positive number so small that when  $|t| \leq \delta$  the point  $\hat{\mathbf{x}}_0 + \hat{\mathbf{a}}t$ , where as ever  $\hat{\mathbf{a}} = (a_1, \dots, a_{n-1})$ , lies in  $\hat{U}_0$ . Then the formulas

$$\hat{\mathbf{x}}(t) = \hat{\mathbf{x}}_0 + \hat{\mathbf{a}}t, \quad x_n(t) = \varphi(\hat{\mathbf{x}}_0 + \hat{\mathbf{a}}t)$$

will define in  $\mathcal{E}$  some curve  $t \mapsto \mathbf{x}(t)$ ,  $|t| \leq \delta$  lying on hypersurface (7) and passing for  $t = 0$  through the point  $\mathbf{x}_0$ . In addition

$$\hat{\mathbf{x}}'(0) = \hat{\mathbf{a}}$$

and

$$\begin{aligned} \mathbf{x}'_n(0) &= \left. \frac{d\varphi(\hat{\mathbf{x}}_0 + \hat{\mathbf{a}}t)}{dt} \right|_{t=0} = \\ &= \frac{\partial \varphi}{\partial x_1}(\hat{\mathbf{x}}_0) a_1 + \dots + \frac{\partial \varphi}{\partial x_{n-1}}(\hat{\mathbf{x}}_0) a_{n-1} = 0, \end{aligned}$$

i.e.  $x'_n(0) = a_n$ . Consequently,  $\mathbf{x}'(0) = \mathbf{a}$  and hence  $\mathbf{a} \in \mathcal{H}_{\mathbf{x}_0}$ .  $\square$

**Definition 6.** The hyperplane of a space  $\mathcal{E}$  passing through a point  $\mathbf{x}_0$  and parallel to the subspace  $\mathcal{H}_{\mathbf{x}_0}$  is called the *tangential hyperplane* of (or to) hyperplane (7) at the point  $\mathbf{x}_0$ .

According to Proposition 1 a tangential hyperplane has the equation

$$(\mathbf{x} - \mathbf{x}_0) \operatorname{grad} F(\mathbf{x}_0) = 0,$$

i.e. the equation

$$\left(\frac{\partial F}{\partial x_1}\right)_0 (x - x_1) + \dots + \left(\frac{\partial F}{\partial x_n}\right)_0 (x - x_n) = 0.$$

The vector  $\operatorname{grad} F(\mathbf{x}_0)$  is orthogonal to the hyperplane.

For  $n = 2$  we obtain the statement known from the course in analysis that in the plane the tangent to an arbitrary curve

$$F(x, y) = 0$$

at its regular point  $(x_0, y_0)$  has the equation

$$\left(\frac{\partial F}{\partial x}\right)_0 (x - x_0) + \left(\frac{\partial F}{\partial y}\right)_0 (y - y_0) = 0.$$

The *length* of a continuous curve (1) is known from the course in analysis to be the limit (if there is one) of the lengths of broken lines inscribed into that curve (we assume the space  $\mathcal{E}$  to be Euclidean). For a smooth curve (1) this limit always exists (the curve is said to be *rectifiable*) and is expressed by the integral

$$(12) \quad \int_a^b |\mathbf{x}'(t)| dt.$$

As a matter of fact the definition of length as the limit of the lengths of inscribed broken lines is never recalled (at least for smooth curves) and only formula (12) is used. The simplest thing therefore is to accept integral (12) as the definition of the length of a smooth curve and to consider the reasoning involving broken lines as the definition's heuristic motivation. This is the way in which we proceed in the third

semester's lectures in similar but more involved situations (for example, when defining the area of a surface).

Let

$$(13) \quad s(t) = \int_a^t |\mathbf{x}'(t)| dt$$

be the length of a segment of curve (1) from  $a$  to  $t$ . If curve (1) is regular, then

$$s'(t) = |x'(t)| > 0$$

and therefore a change of parameter  $t \mapsto s(t)$  is possible. Thus *any regular curve is equivalent to a curve whose parameter is an arc length*. These last curves are usually said to be referred to the natural parameter  $s$ .

In what follows we shall always assume as a rule that all the curves considered are referred to the natural parameter. This is of no fundamental significance of course, but substantially simplifies calculations.

Differentiation with respect to  $s$  will be marked with a dot:

$$\dot{\mathbf{x}}(s) = \frac{d\mathbf{x}(s)}{ds}, \quad \ddot{\mathbf{x}}(s) = \frac{d^2\mathbf{x}(s)}{ds^2}, \text{ etc.}$$

According to formula (13), if  $t = s$ , then

$$\int_a^s |\dot{\mathbf{x}}(s)| ds = s$$

(and  $a = 0$ ) from which it follows that

$$|\dot{\mathbf{x}}(s)| = 1 \text{ for all } s.$$

Conversely if  $|\dot{\mathbf{x}}(t)| = 1$  and  $a = 0$ , then  $t = s$ .

**Lemma 1.** *Let  $s \mapsto \mathbf{u}(s)$  be a vector-valued smooth function such that  $|\mathbf{u}(s)| = 1$  for all  $s$ . Then*

$$(14) \quad \mathbf{u}(s) \dot{\mathbf{u}}(s) = 0 \text{ for all } s.$$

*Proof.* It suffices to note that for a scalar product (as well as for a vector one) of vector-valued functions the usual rule for differentiating a product of functions taking on numerical values is valid (since the usual proof remains completely

valid for this case too). Differentiating the equation  $u(s)^2 = 1$  (and cancelling 2) we therefore obtain (14).  $\square$

In particular we see that

$$\dot{\mathbf{x}}(s) \ddot{\mathbf{x}}(s) = 0 \text{ for all } s.$$

We shall make repeated use of this important formula.

Let us consider a particular case of curves in the plane. Rectangular coordinates in the plane will as always be denoted by  $x$  and  $y$  and a radius vector with these coordinates will be designated by the symbol  $\mathbf{r}$  (instead of the symbol  $\mathbf{x}$  used in the general case). In addition, for any curve  $\mathbf{r} = \mathbf{r}(s)$  in the plane (referred to the natural parameter  $s$ ) we shall designate by the symbol  $\mathbf{t}(s)$  the tangent vector of the curve at a point  $\mathbf{r}(s)$ :

$$\mathbf{t}(s) = \dot{\mathbf{r}}(s).$$

According to the foregoing this vector is a unit vector and

$$\mathbf{t}(s) \dot{\mathbf{t}}(s) = 0 \text{ for any } s.$$

**Definition 7.** The length of a vector  $\dot{\mathbf{t}}(s)$  is designated by the symbol  $k(s)$  and called the *curvature* of a curve  $\mathbf{r} = \mathbf{r}(s)$  at a point  $s$ .

Thus

$$k(s) = |\dot{\mathbf{t}}(s)| = \sqrt{\ddot{x}^2(s) + \ddot{y}^2(s)}.$$

The curvature of a curve referred to an arbitrary parameter  $t$  is the curvature of an equivalent curve referred to the natural parameter. The formula for the curvature (which can be obtained by simple but rather awkward calculations using nothing but formulas for differentiation of functions) is rather involved:

$$k = \left| \frac{x''y' - y''x'}{[(x')^2 + (y')^2]^{3/2}} \right|.$$

The number  $k(s)$  may be interpreted as the instantaneous rotation velocity of the unit vector  $\mathbf{t}(s)$ . It is clear that this velocity is the greater the “more curved” is the curve. Hence the term “curvature”.

Sometimes the so-called *relative curvature*  $k_{\text{rel}}$  is considered (in an oriented plane), equal to curvature  $k$  if (with  $k \neq 0$ ) vectors  $\mathbf{t}$  and  $\dot{\mathbf{t}}$  constitute a positively oriented basis of the plane, and to  $-k$  otherwise. We shall need this curvature in Lecture 25.

**Example 1.** If

$$x(s) = x_0 + sl, \quad y(s) = y_0 + sm, \quad \text{where } l^2 + m^2 = 1,$$

i.e. if the curve under consideration is a straight line, then  $\ddot{x}(s) = 0$  and  $\ddot{y}(s) = 0$ . Therefore  $k(s) = 0$  for all  $s$ , i.e., as was to be expected, the *curvature of a straight line is identically zero*.  $\square$

Since linear functions are, as is easily seen, unique functions, whose second derivative is identically zero, the converse is also true, i.e. *a curve whose curvature is identically zero is a straight line* (or its segment).  $\square$

The point  $\mathbf{r}_0 = \mathbf{r}(s_0)$  of a curve  $\mathbf{r} = \mathbf{r}(s)$  is said to be a *point of rectification* if  $k(s_0) = 0$ .

**Example 2.** The parametric equations of a circle of radius  $R$  in the natural parameter  $s$  are obviously of the form

$$x = R \cos \frac{s}{R}, \quad y = R \sin \frac{s}{R}.$$

Since

$$\ddot{x} = -\frac{1}{R} \cos \frac{s}{R}, \quad \ddot{y} = -\frac{1}{R} \sin \frac{s}{R},$$

we have

$$k(s) = \frac{1}{R}.$$

Thus the *curvature of a circle is constant and equal to the inverse of its radius*.  $\square$

The converse is also true: *a curve with constant curvature is a circle* (or a segment of a circle).  $\square$

This follows from the general theorem which states that for any function  $k = k(s)$  (defined and smooth on the interval  $|s| \leq s_0$ ) there exists (if the number  $s_0$  is sufficiently small) a curve  $\mathbf{r} = \mathbf{r}(s)$ ,  $|s| < s_0$  whose curvature is equal to  $k(s)$ , the curve being unique up to congruence. We shall not prove

the theorem now since in our next lecture we shall establish its analogue for any  $n$ .

If  $k(s) \neq 0$ , then the number  $R(s) = \frac{1}{k(s)}$  is defined, called the *radius of curvature* of a curve at a point  $s$ .

A curve  $r = r(s)$  is said to be a *curve of the general type* if there are no points of rectification on it, i.e. if  $k(s) \neq 0$  for all  $s$ . At each point of such a curve a unit vector

$$n(s) = \frac{\dot{t}(s)}{k(s)}$$

directed along the *normal* to the curve (i.e. along the straight line passing through the point of tangency and perpendicular to the tangent) is defined.

For any  $s$  the vectors  $t(s)$  and  $n(s)$  form an orthonormal basis called the *Frenet moving basis* of a given curve.

By definition

$$\dot{t}(s) = k(s) n(s).$$

We find a similar formula for the vector  $\dot{n}(s)$ . Let

$$\dot{n}(s) = \alpha(s) t(s) + \beta(s) n(s)$$

be an expansion of the vector with respect to the vectors of the basis  $t(s), n(s)$ . Since  $t(s) n(s) = 0$  we have  $\dot{t}(s) n(s) + t(s) \dot{n}(s) = 0$  and so  $\alpha(s) = t(s) \dot{n}(s) = -\dot{t}(s) n(s) = -k(s)$ . On the other hand, by Lemma 1  $\beta(s) = n(s) \times \dot{n}(s) = 0$ . This proves that *for any curve of the general type there are formulas*

$$(15) \quad \begin{aligned} \dot{t}(s) &= k(s) n(s), \\ \dot{n}(s) &= -k(s) t(s) \end{aligned}$$

describing the instantaneous rotation of the moving basis under a change of  $s$ .  $\square$

Formulas (15) are called *Frenet's formulas for a plane curve*.

Now let us consider curves in three-dimensional space (with coordinates  $x, y, z$  and radius vector  $r$  of points). For



any curve  $\mathbf{r} = \mathbf{r}(s)$  (referred to the natural parameter) its tangent vector  $\dot{\mathbf{r}}(s)$  will as before be denoted by  $\mathbf{t}(s)$ . The magnitude  $|\dot{\mathbf{t}}(s)|$  of a vector  $\dot{\mathbf{t}}(s)$  for space curves is also called *curvature* and designated by the symbol  $k(s)$  as before. Thus

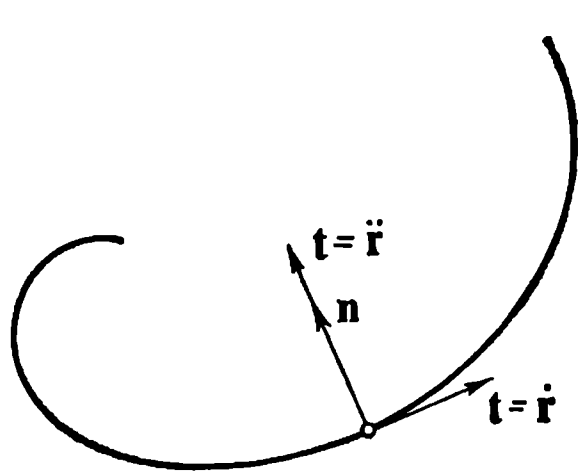
$$k(s) = \sqrt{\ddot{x}(s)^2 + \ddot{y}(s)^2 + \ddot{z}(s)^2}.$$

A curve  $\mathbf{r} = \mathbf{r}(s)$  is said, as in the case  $n = 2$ , to be a *curve of the general type* if  $k(s) \neq 0$  for all  $s$ . For such a curve a unit vector

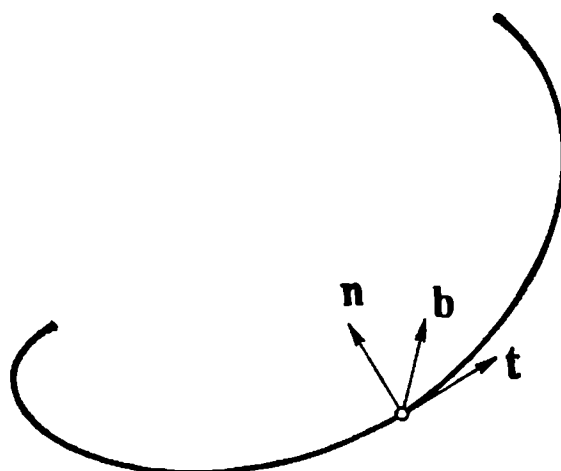
$$\mathbf{n}(s) = \frac{\dot{\mathbf{t}}(s)}{k(s)}$$

called a *vector to the principal normal* to the curve is defined.

But now (assuming the space to be oriented) we can introduce into consideration yet another, a third, vector  $\mathbf{b}(s)$



*Frenet's basis of a plane curve*



*Frenet's basis of a space curve*

constituting together with the vectors  $\mathbf{t}(s)$  and  $\mathbf{n}(s)$  a positively oriented orthonormal basis  $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ . This vector is called the *binormal vector* and the basis  $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ , *Frenet's moving basis* of a given curve of the general type.

By construction (we omit the argument  $s$  to simplify the formulas)

$$\dot{\mathbf{t}} = kn.$$

In addition, since  $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ , we have

$$\dot{\mathbf{b}} = \dot{\mathbf{t}} \times \mathbf{n} + \mathbf{t} \times \dot{\mathbf{n}} = \mathbf{t} \times \dot{\mathbf{n}},$$

whence it follows that  $\dot{\mathbf{b}}\mathbf{t} = 0$ . Since by Lemma 1  $\dot{\mathbf{b}}\mathbf{b} = 0$ , this proves that the vector  $\mathbf{b}$  is collinear with the vector  $\mathbf{n}$ , i.e. there exists a number  $\kappa = \kappa(s)$  such that

$$\dot{\mathbf{b}} = -\kappa\mathbf{n}.$$

The number is called the *torsion* of a given curve at a point  $s$ . It is the rotation velocity of the vector to the binormal.

Differentiating now the equations  $\mathbf{n}\mathbf{t} = 0$  and  $\mathbf{n}\mathbf{b} = 0$  we at once see that  $\dot{\mathbf{n}}\mathbf{t} = -\mathbf{n}\dot{\mathbf{t}} = -k$  and  $\dot{\mathbf{n}}\mathbf{b} = -\mathbf{n}\dot{\mathbf{b}} = \kappa$ . Since in addition  $\dot{\mathbf{n}} = 0$  (Lemma 1) this proves that

$$\dot{\mathbf{n}} = -k\mathbf{t} + \kappa\mathbf{b}.$$

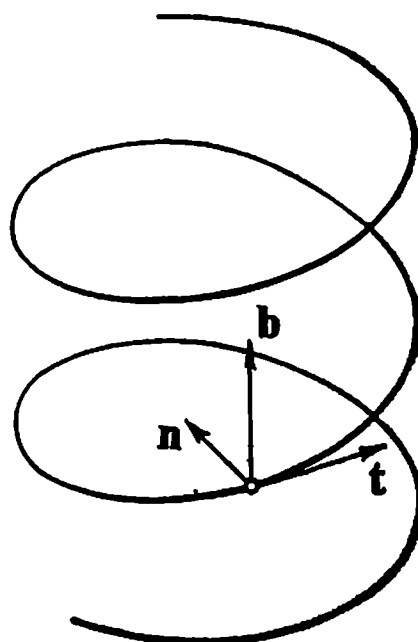
Thus for any general type curve we have the formulas

$$\begin{aligned} \dot{\mathbf{t}} &= k\mathbf{n}, \\ (16) \quad \dot{\mathbf{n}} &= -k\mathbf{t} + \kappa\mathbf{b}, \\ \dot{\mathbf{b}} &= -\kappa\mathbf{n}. \quad \square \end{aligned}$$

These formulas are called *Frenet's formulas for a space curve*.

**Example 1.** If a curve  $\mathbf{r} = \mathbf{r}(s)$  lies in a plane  $\Pi$ , then vectors  $\dot{\mathbf{r}}(s)$  and  $\ddot{\mathbf{r}}(s)$  are parallel to that plane (for this is the case for the increments  $\mathbf{r}(s + \Delta s) - \mathbf{r}(s)$  and  $\dot{\mathbf{r}}(s + \Delta s) - \dot{\mathbf{r}}(s)$  of the vectors  $\mathbf{r}(s)$  and  $\dot{\mathbf{r}}(s)$ ). Therefore  $\mathbf{t}(s), \mathbf{n}(s) \parallel \Pi$  and hence  $\mathbf{b}(s) \perp \Pi$ . This proves that  $\mathbf{b}(s) = \text{const}$  and so  $\kappa(s) = 0$  for all  $s$ . Conversely, let  $\kappa(s) = 0$  for all  $s$  and hence  $\mathbf{b}(s) = \mathbf{b}_0 = \text{const}$ . Then  $(\mathbf{r}(s) \mathbf{b}_0)' = \mathbf{t}(s) \mathbf{b}_0 = 0$  for all  $s$  and therefore  $\mathbf{r}(s) \mathbf{b}_0 = \text{const}$ . This means that the curve  $\mathbf{r} = \mathbf{r}(s)$  lies in the plane  $\mathbf{r}\mathbf{b}_0 = \text{const}$ . Thus a curve in space is a plane curve if and only if its torsion is identically zero.  $\square$

**Example 2.** A *circular helix* is the path described by a point moving at a constant velocity along a generator of a



A circular helix

right circular cylinder rotating uniformly about its axis. The equations of the helix are of the form

$$x = a \cos t, \quad y = a \sin t, \quad z = bt.$$

We have

$$x' = -a \sin t, \quad y' = a \cos t, \quad z' = b,$$

whence

$$s' = \sqrt{(x')^2 + (y')^2 + (z')^2} = \sqrt{a^2 + b^2}.$$

Thus  $s = ct$ , where  $c = \sqrt{a^2 + b^2}$  and hence

$$x = a \cos \frac{s}{c}, \quad y = a \sin \frac{s}{c}, \quad z = \frac{b}{c} s.$$

Since

$$\begin{aligned} \dot{x} &= -\frac{a}{c} \sin \frac{s}{c}, & \dot{y} &= \frac{a}{c} \cos \frac{s}{c}, & \dot{z} &= \frac{b}{c}, \\ \ddot{x} &= -\frac{a}{c^2} \cos \frac{s}{c}, & \ddot{y} &= -\frac{a}{c^2} \sin \frac{s}{c}, & \ddot{z} &= 0, \end{aligned}$$

we have

$$k = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2} = \frac{a}{c^2} = \text{const}$$

and

$$\mathbf{t} = -\frac{a}{c} \sin \frac{s}{c} \mathbf{i} + \frac{a}{c} \cos \frac{s}{c} \mathbf{j} + \frac{b}{c} \mathbf{k},$$

$$\mathbf{n} = -\cos \frac{s}{c} \mathbf{i} - \sin \frac{s}{c} \mathbf{j},$$

$$\mathbf{b} = \mathbf{t} \times \mathbf{n} =$$

$$\begin{aligned} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{a}{c} \sin \frac{s}{c} & \frac{a}{c} \cos \frac{s}{c} & \frac{b}{c} \\ -\cos \frac{s}{c} & -\sin \frac{s}{c} & 0 \end{vmatrix} = \frac{b}{c} \sin \frac{s}{c} \mathbf{i} - \\ &\quad -\frac{b}{c} \cos \frac{s}{c} \mathbf{j} + \frac{a}{c} \mathbf{k}. \end{aligned}$$

Therefore

$$\dot{\mathbf{b}} = \frac{b}{c^2} \cos \frac{s}{c} \mathbf{i} + \frac{b}{c^2} \sin \frac{s}{c} \mathbf{j} = -\frac{b}{c^2} \mathbf{n}$$

and so

$$\kappa = \frac{b}{c^2} = \text{const.}$$

Thus the *curvature and the torsion of a circular helix are constant*.  $\square$

According to a general theorem, which we are going to prove in the next lecture, and conversely, *every curve whose curvature and torsion are constant is a circular helix (or its arc)*.

# Lecture 24

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*Projections of a curve onto the coordinate planes of the moving  $n$ -hedron. Frenet's formulas for a curve in  $n$ -dimensional space. Representation of a curve by its curvatures. Regular surfaces. Examples of surfaces*

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To investigate the behaviour of an arbitrary space curve  $\mathbf{r} = \mathbf{r}(s)$  near some of its points we choose the origin  $O$  in that point, choose vectors  $\mathbf{t}_0$ ,  $\mathbf{n}_0$ ,  $\mathbf{b}_0$  of the moving  $n$ -hedron in the point  $O$  to be the vectors of the basis  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  and count the natural parameter  $s$  off from  $O$ . Then

$$\begin{aligned}\mathbf{r}(0) &= 0, \quad \dot{\mathbf{r}}(0) = \mathbf{t}_0 = \mathbf{i}, \quad \ddot{\mathbf{r}}(0) = k_0 \mathbf{n}_0 = k_0 \mathbf{j}, \\ \dddot{\mathbf{r}}(0) &= (\dot{\mathbf{k}})_0 \mathbf{n}_0 + k_0 \dot{\mathbf{n}}_0 = -k_0^2 \mathbf{i} + (\dot{k})_0 \mathbf{j} + k_0 \kappa_0 \mathbf{k},\end{aligned}$$

where  $k_0$ ,  $(\dot{k})_0$  and  $\kappa_0$  are the values of functions  $k$ ,  $\dot{k}$  and  $\kappa$  for  $s = 0$ . Hence, using the Taylor formula

$$\begin{aligned}\mathbf{r}(s) &= \mathbf{r}(0) + s\dot{\mathbf{r}}(0) + \frac{s^2}{2}\ddot{\mathbf{r}}(0) + \frac{s^3}{6}\dddot{\mathbf{r}}(0) + \dots = \\ &= \left(s - \frac{k_0^2}{6}s^3 + \dots\right)\mathbf{i} + \left(\frac{k_0}{2}s^2 + \frac{(\dot{k})_0}{6}s^3 + \dots\right)\mathbf{j} + \\ &\quad + \left(\frac{k_0\kappa_0}{6}s^3 + \dots\right)\mathbf{k}.\end{aligned}$$

This implies that near the point  $O$  our curve is given by the parametric equations

$$\begin{aligned}x &= s + \dots, \\ y &= \frac{k_0}{2}s^2 + \dots, \\ z &= \frac{k_0\kappa_0}{6}s^3 + \dots\end{aligned}$$

If  $k_0 \neq 0$ ,  $\kappa_0 \neq 0$ , then the projection of the curve onto the plane  $Oij = Ot_0n_0$  (incidentally, this plane is called the *osculating plane* of the curve at the point  $O$ ) approximately coincides with the parabola

$$x = s, \quad y = \frac{k_0}{2} s^2;$$

its projection onto the plane  $Ojk = On_0b_0$  (called the *normal plane* of the curve at the point  $O$ ) does with the semi-cubical parabola

$$y = \frac{k_0}{2} s^2, \quad z = \frac{k_0\kappa_0}{6} s^3$$

and finally its projection onto the plane  $Oik = Ot_0b_0$  (called a *rectifying plane* of the curve at the point  $O$ ) does with the cubical parabola

$$x = s, \quad z = \frac{k_0\kappa_0}{2} s^3.$$

This gives a fairly clear idea of how a space curve is constructed near any of its points (at which curvature and torsion are different from zero).

We now extend the results obtained in the preceding lecture to include the case of an arbitrary  $n$ .

Let  $\mathbf{x} = \mathbf{x}(s)$ ,  $|s| \leq s_0$  be an arbitrary curve (referred to the natural parameter) in an  $n$ -dimensional oriented Euclidean space  $\mathcal{E}$ . Assuming that for any  $s$  the vectors

$$\dot{\mathbf{x}}(s), \dots, \mathbf{x}^{(n-1)}(s)$$

are linearly independent (such curves are called *curves of the general type*) and applying to those vectors the Gram-Schmidt orthogonalization process we obtain an orthonormal family of vectors  $\mathbf{t}_1(s), \dots, \mathbf{t}_{n-1}(s)$ . Let  $\mathbf{t}_n(s)$  be a vector (uniquely defined) extending that family to a positively oriented orthonormal basis

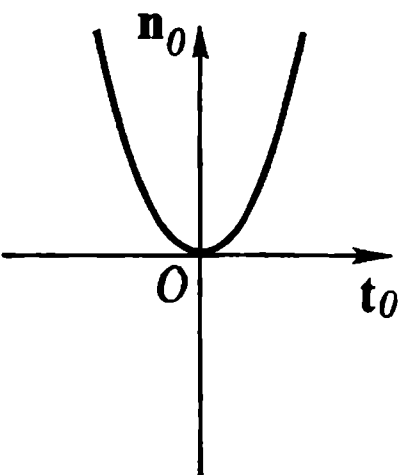
$$(1) \quad \mathbf{t}_1(s), \dots, \mathbf{t}_{n-1}(s), \mathbf{t}_n(s).$$

**Definition 1.** Basis (1) is called *Frenet's moving basis* of a curve  $\mathbf{x} = \mathbf{x}(s)$  of the general type at a point  $s$ .

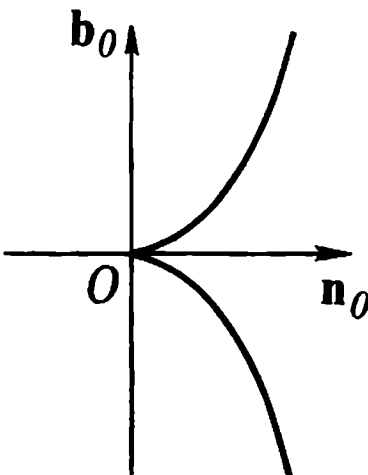
Let

$$\dot{\mathbf{t}}_i = \sum_{j=1}^n \alpha_{ij} \mathbf{t}_j \quad i = 1, \dots, n$$

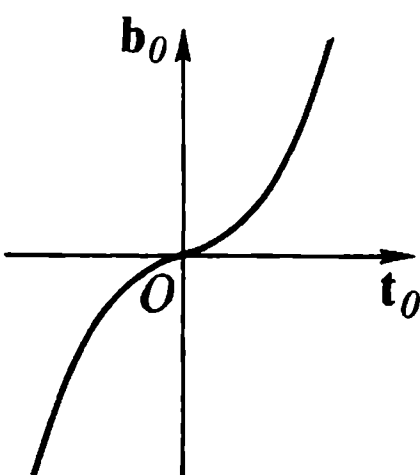
(we omit the argument  $s$  to simplify the formulas). Since by construction the vector  $\mathbf{t}_i$ ,  $i = 1, \dots, n - 1$ , is linearly



*Projection onto the os-  
culating plane*



*Projection onto the nor-  
mal plane*



*Projection onto a rec-  
tifying plane*

expressible in terms of vectors  $\dot{\mathbf{x}}, \dots, \overset{(i)}{\mathbf{x}}$ , the vector  $\dot{\mathbf{t}}_i$  is linearly expressible in terms of vectors  $\dot{\mathbf{x}}, \dots, \overset{i+1}{\mathbf{x}}$ . Since the last vectors are linearly expressible in terms of the vectors  $\mathbf{t}_1, \dots, \mathbf{t}_{i+1}$ , this proves that  $\alpha_{ij} = 0$  provided  $j > i + 1$ .

On the other hand, since  $\mathbf{t}_i \mathbf{t}_j = \delta_{ij}$ , we have  $\dot{\mathbf{t}}_i \mathbf{t}_j + \mathbf{t}_i \dot{\mathbf{t}}_j = 0$ , i.e.

$$\alpha_{ij} + \alpha_{ji} = 0.$$

Therefore  $\alpha_{ii} = 0$  and  $\alpha_{ij} = 0$  provided  $j < i - 1$ . Thus only the coefficients  $\alpha_{i,i+1} = -\alpha_{i+1,i}$  can be non-zero. Setting

$$k_1 = \alpha_{12}, k_2 = \alpha_{23}, \dots, k_{n-1} = \alpha_{n-1,n}$$







of vector-valued functions  $\mathbf{t}_1 = \mathbf{t}_1(s), \dots, \mathbf{t}_n = \mathbf{t}_n(s)$  on the interval  $|s| \leq s_0$  such that

- (i) for any  $s$  there are relations (2);
- (ii) for  $s = 0$  there are

$$(7) \quad \mathbf{t}_1(0) = \mathbf{i}_1, \dots, \mathbf{t}_n(0) = \mathbf{i}_n.$$

**Stage 2.** We consider scalar products  $\mathbf{t}_i \mathbf{t}_j$ ,  $i, j = 1, \dots, n$ . According to relations (2), for these products we have

$$((\mathbf{t}_i \mathbf{t}_j))' = \dot{\mathbf{t}}_i \mathbf{t}_j + \mathbf{t}_i \dot{\mathbf{t}}_j = (-k_{i-1} \mathbf{t}_{i-1} + k_i \mathbf{t}_{i+1}) \mathbf{t}_j + \mathbf{t}_i (-k_{j-1} \mathbf{t}_{j-1} + k_j \mathbf{t}_{j+1})$$

(we assume by convention that  $\mathbf{t}_0 = 0$  and  $\mathbf{t}_{n+1} = 0$ ), i.e. the equations

$$(8) \quad (\mathbf{t}_i \mathbf{t}_j)' = -k_{i-1} (\mathbf{t}_{i-1} \mathbf{t}_j) + k_i (\mathbf{t}_{i+1} \mathbf{t}_j) - k_{j-1} (\mathbf{t}_i \mathbf{t}_{j-1}) + k_j (\mathbf{t}_i \mathbf{t}_{j+1})$$

which may be regarded as equations of the form (6) for  $m = \frac{n(n+1)}{2}$  functions  $\mathbf{t}_i \mathbf{t}_j$ . By the EUS theorem therefore there exists only one set of these functions possessing the property that for  $s = 0$  they are equal to  $\delta_{ij} = \mathbf{i}_i \mathbf{i}_j$  (i.e. to zero if  $i \neq j$  and to unity if  $i = j$ ).

On the other hand, a direct check shows that equations (8) satisfy the functions  $\mathbf{t}_i \mathbf{t}_j$  identically equal to  $\delta_{ij}$ . (Indeed, when  $i \neq j-1, j+1$  all the terms of the sum  $-k_{i-1} \delta_{i-1,j} + k_i \delta_{i+1,j} - k_{j-1} \delta_{i,j-1} + k_j \delta_{i,j+1}$  are zero and when  $i = j-1, i+1$  the sum has only two nonzero but mutually cancelled terms.) Hence for all  $s$  there are by virtue of the EUS theorem equations  $\mathbf{t}_i \mathbf{t}_j = \delta_{ij}$ ,  $i, j = 1, \dots, n$ , implying that for any  $s$ ,  $|s| \leq s_0$ , the vectors  $\mathbf{t}_1, \dots, \mathbf{t}_n$  constitute an orthonormal basis.

Since for  $s = 0$  that basis coincides with a positively oriented basis  $\mathbf{i}_1, \dots, \mathbf{i}_n$ , the basis  $\mathbf{t}_1, \dots, \mathbf{t}_n$  is positively oriented for any  $s$  too.

**Stage 3.** We compose consecutive derivatives of the vector  $\mathbf{t}_1$ :

$$(9) \quad \mathbf{t}_1, \dot{\mathbf{t}}_1, \ddot{\mathbf{t}}_1, \dots, \overset{(n-1)}{\mathbf{t}}_1,$$

and apply to them the Gram-Schmidt orthogonalization process. Since the vector  $\mathbf{t}_1$  is a unit vector, we need not do anything in the first step of the process. Since the vector  $\dot{\mathbf{t}}_1$  is orthogonal to the vector  $\mathbf{t}_1$  (by Lemma 1 of the preceding lecture), in the second step we must only normalize it. Since according to what has been proved the vector  $\mathbf{t}_2$  is a unit vector and  $k_1 > 0$  by the hypothesis, according to the first of the relations (2)  $|\dot{\mathbf{t}}_1| = k_1$ . In the second step therefore we obtain the vector

$$\mathbf{t}_2 = \frac{\dot{\mathbf{t}}_1}{k_1}.$$

In the third step we should consider the vector

$$\ddot{\mathbf{t}}_1 = (k_1 \mathbf{t}_2)^\cdot = \dot{k}_1 \mathbf{t}_2 + k_1 \dot{\mathbf{t}}_2 = -k_1^2 \mathbf{t}_1 + \dot{k}_1 \mathbf{t}_2 + k_1 k_2 \mathbf{t}_3,$$

subtract from it the linear combination of vectors  $\mathbf{t}_1$  and  $\mathbf{t}_2$  to obtain a vector orthogonal to those vectors and then normalize the vector. But since according to what has been proved the vectors  $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$  constitute an orthonormal family and by the hypothesis  $k_1 k_2 > 0$ , the result of this procedure is obviously the vector  $\mathbf{t}_3$ .

It is clear that this reasoning is of a general character so that at each step of the orthogonalization process we obtain the corresponding vector  $\mathbf{t}_i$ ,  $i = 1, \dots, n-1$ . This proves that the family of vectors  $\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_{n-1}$  is uniquely characterized as an orthonormal family of vectors obtained from family (9) by the Gram-Schmidt orthogonalization process.

**Stage 4.** Let

$$(10) \quad \mathbf{x}(s) = \int_0^s \mathbf{t}_1(s) ds, \quad |s| \leq s_0.$$

Then  $\mathbf{x}(0) = 0$  and  $\dot{\mathbf{x}}(s) = \mathbf{t}_1(s)$ , i.e. the curve  $\mathbf{x} = \mathbf{x}(s)$ ,  $|s| \leq s_0$ , begins at the point  $O$  and has at a point  $\mathbf{x}(s)$  the tangent vector  $\mathbf{t}_1(s)$ . But for every curve the first  $n-1$  vectors of the moving basis are vectors obtained from the first  $n-1$  derivatives of the tangent vector by the Gram-Schmidt orthogonalization process. According to the fore-

going therefore those vectors coincide with the vectors  $t_1, \dots, t_{n-1}$ .

As to the last vector of the moving basis, it is uniquely characterized as unit vector constituting together with the first  $n - 1$  vectors a positively oriented basis. Since the basis  $t_1, \dots, t_{n-1}, t_n$  was seen to be positively oriented, that vector must be the vector  $t_n$ .

Thus we have proved that for any  $s$  the vectors  $t_1(s), \dots, \dots, t_n(s)$  constitute the moving basis of the curve  $x = x(s)$ . Since for these vectors we have Frenet's formulas (2), the functions  $k_i(s)$ ,  $i = 1, \dots, n - 1$ , appearing in the formulas must be the curvatures of the curve  $x = x(s)$ .

This completes the proof of the existence of a curve  $x = x(s)$  possessing properties (i) and (ii).

The uniqueness of the curve follows from the fact that according to the EUS theorem the moving basis  $t_1(s), \dots, \dots, t_n(s)$  is uniquely defined by equations (2) and the initial conditions (7) and the radius vector  $x(s)$  is uniquely defined (by formula (10)) by the relation  $\dot{x}(s) = t_1(s)$  and the initial condition  $x(0) = 0$ .  $\square$

By analogy with Definitions 1 to 3 of Lecture 22, for any  $k$ ,  $0 < k < n$ , a "parametric" definition can be given of a  $k$ -dimensional surface in  $n$ -dimensional space. For simplicity we confine ourselves to the case where  $k = 2$  and  $n = 3$ .

Let  $W$  be an arbitrary open set in the two-dimensional space  $\mathbb{R}^2$  whose points are pairs  $(u, v)$  of real numbers. An arbitrary mapping  $W \rightarrow \mathcal{E}$  of that set into a three-dimensional Euclidean space  $\mathcal{E}$  is given (if an origin  $O$  is chosen in  $\mathcal{E}$ ) either by a vector-valued function  $r = r(u, v)$  defined in  $W$  or (if rectangular coordinates  $x, y, z$  are introduced in  $\mathcal{E}$ ) by three numerical functions

$$(11) \quad x = x(u, v), \quad y = y(u, v), \quad z = z(u, v).$$

As before, we shall consider only *smooth* mappings  $(u, v) \mapsto r(u, v)$ , i.e. such that functions (11) are smooth in  $W$ . The partial derivatives

$$(12) \quad r_u = x_u \mathbf{i} + y_u \mathbf{j} + z_u \mathbf{k}, \quad r_v = x_v \mathbf{i} + y_v \mathbf{j} + z_v \mathbf{k}$$

will therefore be defined (we omit the arguments  $u, v$  for simplicity).

**Definition 2.** A mapping  $(u, v) \mapsto \mathbf{r}(u, v)$  is said to be a *regular surface* if for any point  $(u, v) \in W$  vectors (12) are linearly independent.

The set of all points of  $\mathcal{E}$  whose radius vectors are of the form  $\mathbf{r}(u, v)$ ,  $(u, v) \in W$ , is called the *support* of the surface  $\mathbf{r} = \mathbf{r}(u, v)$ .

Recall from the course in analysis that the bijective mapping  $W \rightarrow W_1$  of an open set  $W \subset \mathbb{R}^2$  onto an open set  $W_1 \subset \mathbb{R}^2$  is said to be a *diffeomorphism* if the functions

$$(13) \quad u_1 = u_1(u, v), \quad v_1 = v_1(u, v)$$

and

$$(14) \quad u = u(u_1, v_1), \quad v = v(u_1, v_1)$$

giving the mapping  $W \rightarrow W_1$  and the inverse transformation  $W_1 \rightarrow W$  are smooth functions. For the bijective mapping given by smooth functions (13) to be a diffeomorphism it is necessary and sufficient that its *Jacobian*

$$(15) \quad \frac{\partial(u_1, v_1)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial u_1}{\partial u} & \frac{\partial u_1}{\partial v} \\ \frac{\partial v_1}{\partial u} & \frac{\partial v_1}{\partial v} \end{vmatrix}$$

should be nonzero everywhere in domain  $W$ . If, on the other hand, Jacobian (15) of functions (13) (which *a priori* are not assumed to give a bijection) is nonzero everywhere in  $W$ , then the mapping they give is a *local diffeomorphism*, i.e. for any point  $(u_0, v_0) \in W$  there exists a neighbourhood  $U \subset W$  in which the mapping is its diffeomorphism onto some neighbourhood  $U_1 \subset W_1$  of a point  $(u_1(u_0, v_0), v_1(u_0, v_0))$  (this is the so-called *inverse transform theorem*).

Now let us be given two surfaces:

$$(16) \quad \mathbf{r} = \mathbf{r}(u, v), \quad (u, v) \in W,$$

and

$$(17) \quad \mathbf{r} = \mathbf{r}_1(u_1, v_1), \quad (u_1, v_1) \in W_1.$$

**Definition 3.** Surfaces (16) and (17) are said to be *equivalent* if there exists a diffeomorphism

$$u_1 = u_1(u, v), \quad v_1 = v_1(u, v)$$

of an open set  $W$  onto an open set  $W_1$  such that

$$\mathbf{r}(u, v) = \mathbf{r}_1(u_1(u, v), v_1(u, v))$$

for any point  $(u, v) \in W$ .

It is clear that equivalent surfaces have the same support.

Any smooth function  $z = z(x, y)$  of two variables defined in domain  $W$  gives by the formula

$$\mathbf{r}(u, v) = u\mathbf{i} + v\mathbf{j} + z(u, v)\mathbf{k}$$

a regular surface called the *graph* of that function.

It turns out that with an appropriate choice of coordinate axes *any regular surface is locally equivalent to the graph of some smooth function*. Indeed, since vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$  are linearly independent, at any point  $(u_0, v_0) \in W$  the rank of the matrix

$$\begin{pmatrix} x_u & y_u & z_u \\ x_v & y_v & z_v \end{pmatrix}$$

equals two, i.e. at least one of its minors of the second order is nonzero. For definiteness let

$$\begin{vmatrix} x_u & y_u \\ x_v & y_v \end{vmatrix} \neq 0.$$

Then, by the inverse transform theorem (applied to functions  $x = x(u, v)$  and  $y = y(u, v)$ ), there will exist a neighbourhood  $U_0 \subset W$  of the point  $(u_0, v_0)$  and a neighbourhood  $V_0 \subset \mathbb{R}^2$  of a point  $(x_0, y_0) \in \mathbb{R}^2$ , where  $x_0 = x(u_0, v_0)$ ,  $y_0 = y(u_0, v_0)$ , such that the functions  $x = x(u, v)$  and  $y = y(u, v)$  effect a diffeomorphism of the neighbourhood  $U_0$  onto the neighbourhood  $V_0$ . Then if,

$$u = u(x, y), \quad v = v(x, y)$$

are the functions effecting the inverse diffeomorphism, in the neighbourhood  $U_0$  the surface  $\mathbf{r} = \mathbf{r}(u, v)$  will be equivalent to the graph of the function  $z = z(u(x, y), v(x, y))$ .  $\square$

Although a surface is not a set, terminologically it is often identified with its support. Thus, for example, points of the support of a surface are called *points of the surface* and so on.

In general a regular surface may be a noninjective mapping into  $\mathcal{E}$  (it may have points, curves and even entire domain of “self-intersection”) but in this semester’s lectures we shall concern ourselves only with sufficiently small domains of it in which it is equivalent to the graph and hence is an *injection*.

If a point  $M$  of a surface has a radius vector  $\mathbf{r}(u, v)$ , then the numbers  $u$  and  $v$  are said to be the *coordinates* of that point *on the surface*. By virtue of injectivity of the mapping  $(u, v) \mapsto \mathbf{r}(u, v)$  this definition is correct.

Any curve

$$(18) \quad u = u(t), \quad v = v(t)$$

in domain  $W$  determines a curve

$$(19) \quad \mathbf{r} = \mathbf{r}(u(t), v(t))$$

in  $\mathcal{E}$  which is said *to lie on the surface*  $\mathbf{r} = \mathbf{r}(u, v)$ . Equations (18) are called the equations of curve (19) in coordinates  $u, v$  on the surface.

In particular, defined on the surface are curves  $u = \text{const}$  and  $v = \text{const}$ . These are called *coordinate curves* and their collection is called the *coordinate network* on the surface.

### *Examples of surfaces.*

#### 1. The support of the surface

$$(20) \quad x = R \cos u, \quad y = R \sin u, \quad z = v$$

is a right circular cylinder of radius  $R$ . Accordingly surface (20) is also called a (circular) *cylinder*.

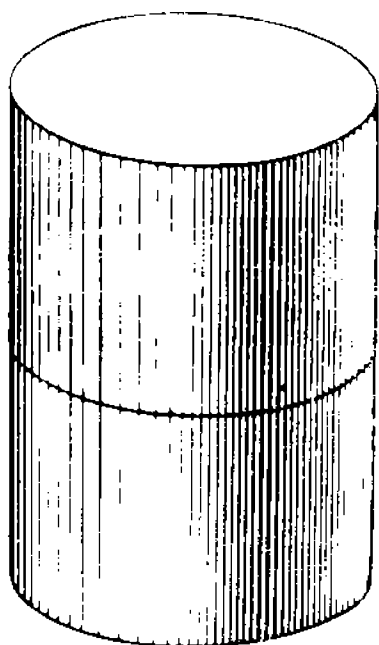
When  $-\infty < u < +\infty$  each point of the cylinder is covered an infinite (countable) number of times by the points of the plane  $(u, v)$ . To attain injectivity it should be assumed that  $0 < u < 2\pi$ , but then a “slotted” cylinder results. All our considerations being local, we shall ignore such situations in what follows.

The coordinate network on a cylinder consists of “vertical” straight lines  $u = \text{const}$  and “horizontal” circles  $v = \text{const}$ .

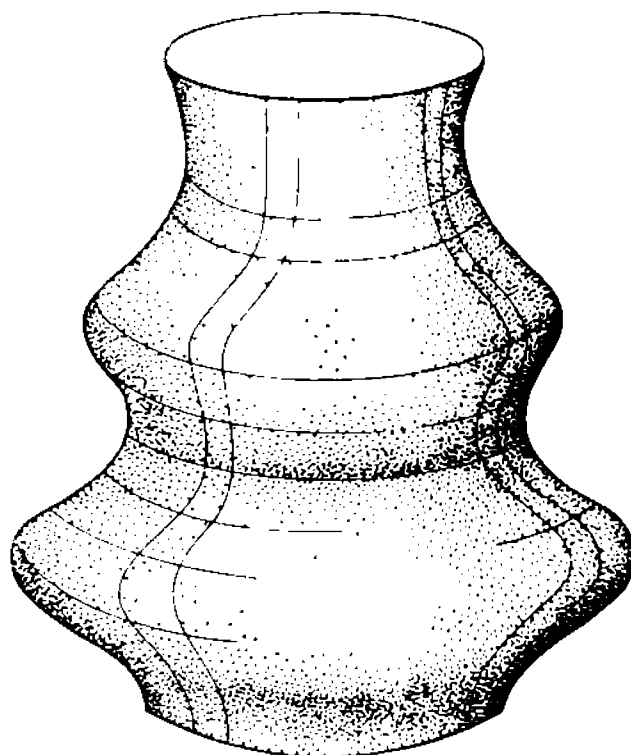
2. Let  $x = x(v)$ ,  $z = z(v)$  be an arbitrary regular curve on the plane  $Oxz$  not intersecting the axis  $Oz$ . The surface

$$(21) \quad x = x(v) \cos u, \quad y = x(v) \sin u, \quad z = z(v)$$

is called a *surface of revolution* and the curve  $x = x(v)$ ,  $z = z(v)$  is its *profile*. Intuitively, surface (21) is obtained by rotating its profile about the axis  $Oz$ .



*A circular cylinder*



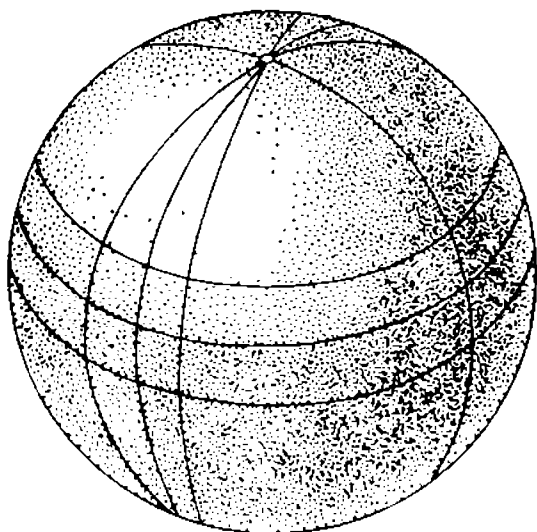
*A surface of revolution*

The regularity of surface (21), i.e. linear independence of vectors

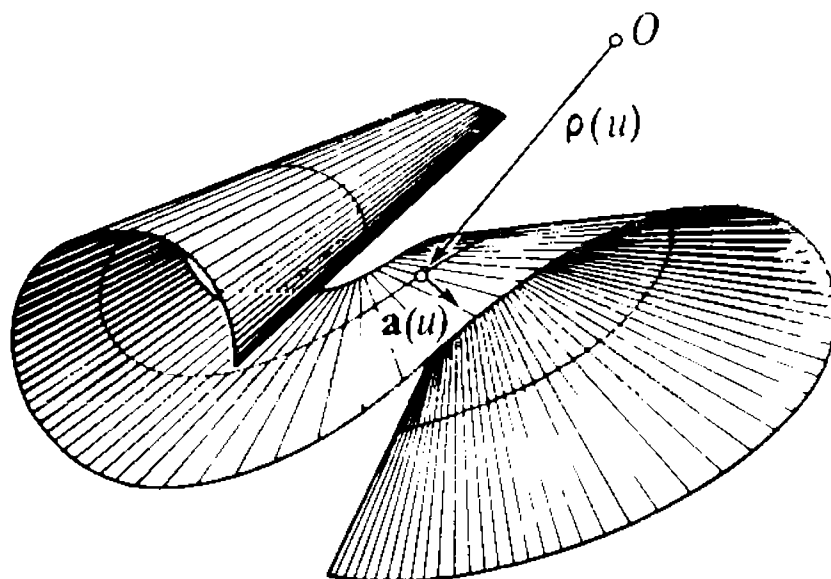
$$\mathbf{r}_u = (-x(v) \sin u, x(v) \cos u, 0)$$

$$\mathbf{r}_v = (x'(v) \cos u, x'(v) \sin u, z'(v))$$

is ensured by the regularity of the profile (i.e. by the condi-



*A sphere*

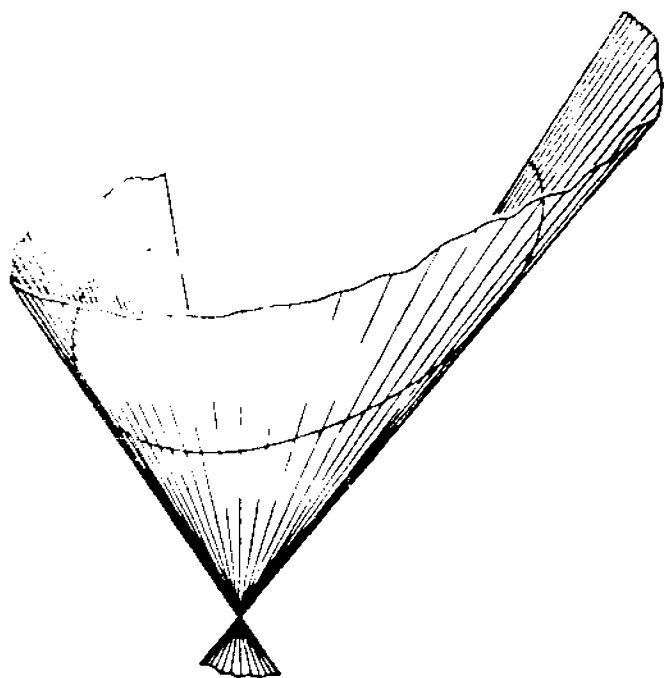


*A ruled surface*

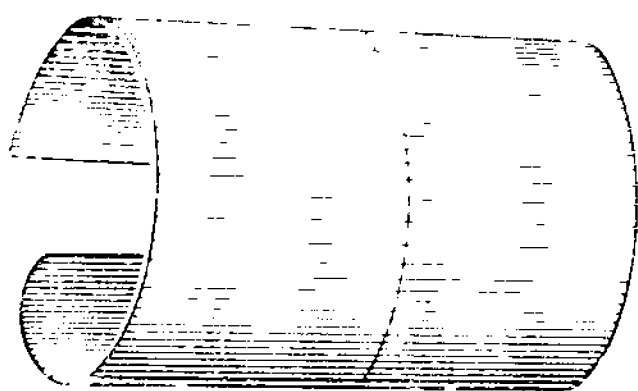
tion  $x'(v)^2 + z'(v)^2 \neq 0$ ) and by the fact that the profile does not intersect the rotation axis  $Oz$  (i.e. by  $x(v) \neq 0$ ).



The coordinate network on the surface (21) consists of curves which are rotations of the profile about the axis  $Oz$  (they are called *meridians*) and circles perpendicular to them (*parallels*).



*A cone*



*A cylinder*

A cylinder is a surface of revolution whose profile is a straight line  $x = R$ ,  $z = v$ .

A surface of revolution with profile  $x = R \cos v$ ,  $z = R \sin v$  (a circle) is the sphere

$$x = R \cos v \cos u, y = R \cos v \sin u, z = R \sin v$$

of radius  $R$  with centre at a point  $O$ . Coordinates  $u$  and  $v$  are the well-known "geographical coordinates", longitude and latitude, and the coordinate curves are geographical meridians and parallels.

Note that strictly speaking we must consider only the portion of the circle  $x = R \cos v$ ,  $z = R \sin v$  that does not intersect the axis  $Oz$  and hence only the corresponding portion of the sphere (a "pole-punctured sphere"). This is reflected in the fact that coordinates  $u$  and  $v$  become meaningless at the poles. We have already agreed above, however, to ignore such phenomena.

3. A surface  $\mathbf{r} = \mathbf{r}(u, v)$  is said to be a *ruled* surface if

$$(22) \quad \mathbf{r}(u, v) = \boldsymbol{\rho}(u) + v\mathbf{a}(u),$$

where  $\boldsymbol{\rho}(u)$  and  $\mathbf{a}(u)$  are arbitrary vector-valued functions possessing the property (ensuring regularity) that the vectors  $\boldsymbol{\rho}'(u) + v\mathbf{a}'(u)$  and  $\mathbf{a}(u)$  are linearly independent for all  $u$  and  $v$  considered (so that, in particular,  $\mathbf{a}(u) \neq 0$  for all  $u$ ). A coordinate curve  $u = u_0 = \text{const}$  is a straight

line, with direction vector  $\mathbf{a}(u_0)$ , passing through the point with radius vector  $\boldsymbol{\rho}(u_0)$ . Thus, intuitively, a ruled surface is swept out by a straight line moving in space. Cf. Definition 1 of Lecture 23 in [1].

It is clear that without loss of generality we may assume the vector  $\mathbf{a}(u)$  to be a unit vector:

$$\mathbf{a}^2(u) = 1 \text{ for all } u.$$

If  $\boldsymbol{\rho}'(u) = 0$  for all  $u$ , i.e.  $\boldsymbol{\rho}(u) = \text{const}$ , then, after translation of the origin, we obtain instead of (22) an equation of the form

$$(23) \quad \mathbf{r} = v\mathbf{a}(u).$$

It is a cone whose directrix is a regular space curve  $\mathbf{a} = \mathbf{a}(u)$ .

If  $\mathbf{a}'(u) = 0$  for all  $u$ , i.e.  $\mathbf{a}(u) = \text{const}$ , then surface (22) is a cylinder with directrix  $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$  (a space one in general).

If the vector  $\boldsymbol{\rho}'$  is not identically zero, then, going if necessary to a smaller domain in  $\mathbb{R}^2$ , we may assume that  $\boldsymbol{\rho}'(u) \neq 0$  for all  $u$ . Then  $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$  is a regular curve in space and we may assume that  $u$  is the natural parameter (arc length) on that curve. Cone (23) may also be given by an equation of form (22) with  $\boldsymbol{\rho}'(u) \neq 0$ . To do this it is sufficient to put  $\boldsymbol{\rho}(u) = \mathbf{a}(u)$  in (22) (if  $\mathbf{a}'(u) \neq 0$  of course).

If  $\mathbf{a}(u)$  is the tangent vector  $\boldsymbol{\tau}(u)$  of a curve  $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$ , then surface (22) is said to be a *surface of tangents*. Similarly defined are a *surface of principal normals* and a *surface of binormals*.

If a curve  $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$  is a plane curve, then its surface of binormals is a cylinder over that curve.

# Lecture 25

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*Vectors tangential to a surface. The tangential plane. The first quadratic form of a surface. Mensuration of lengths and angles on a surface. Diffeomorphisms of surfaces. Isometries and the intrinsic geometry of a surface. Examples. Developables*

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By analogy with Definition 5 of Lecture 22 the *tangent vector* to a (regular) surface

$$(1) \quad \mathbf{r} = \mathbf{r}(u, v), \quad (u, v) \in W \subset \mathbb{R}^2$$

at a point  $(u_0, v_0)$  is the tangent vector of an arbitrary curve on a surface passing through the point  $(u_0, v_0)$ . Since locally a surface is the graph of a smooth function, this definition actually coincides with Definition 5 of Lecture 22 (i.e. gives the same vectors). According to Proposition 1 of Lecture 23 therefore the *collection of all the tangent vectors of surface (1) at a given point  $(u_0, v_0)$  is a two-dimensional vector space.*  $\square$

However, this fact is easy to prove directly as well. Indeed, any curve on surface (1) passing at  $t = t_0$  through a point  $(u_0, v_0)$  is given as a curve in space by a vector function of the form

$$(2) \quad \mathbf{r}(t) = \mathbf{r}(u(t), v(t)), \quad t_1 \leq t \leq t_2,$$

where  $u = u(t)$  and  $v = v(t)$  are smooth functions such that  $u(t_0) = u_0$  and  $v(t_0) = v_0$ . Therefore

$$(3) \quad \mathbf{r}'(t) = u'(t) \mathbf{r}_u + v'(t) \mathbf{r}_v$$

and in particular

$$\mathbf{r}'(t_0) = u'(t_0) (\mathbf{r}_u)_0 + v'(t_0) (\mathbf{r}_v)_0.$$

Thus any tangent vector to surface (1) at a point  $(u_0, v_0)$  is a linear combination of vectors  $(\mathbf{r}_u)_0$  and  $(\mathbf{r}_v)_0$  (noncollinear ones by the hypothesis). Conversely, if

$$(4) \quad \mathbf{c} = a (\mathbf{r}_u)_0 + b (\mathbf{r}_v)_0,$$

where  $a$  and  $b$  are arbitrary numbers, then  $\mathbf{c} = \mathbf{r}'(t_0)$ , where  $\mathbf{r}(t) = \mathbf{r}(u_0 + a(t - t_0), v_0 + b(t - t_0))$ , and hence  $\mathbf{c}$  is a vector tangential to surface (1) at a point  $(u_0, v_0)$ .

This completes the proof, since vectors (4) constitute a two-dimensional vector space.  $\square$

**Definition 1.** A vector space consisting of vectors (4) is called the *tangent plane* to surface (1) at a point  $(u_0, v_0)$ .

The same term is applied also to the corresponding plane in space passing through a point  $\mathbf{r}(u_0, v_0)$ . The plane has a direction bivector  $(\mathbf{r}_u)_0 \wedge (\mathbf{r}_v)_0$  and is therefore given in coordinates  $x, y, z$  by the equation

$$\begin{vmatrix} x - x(u_0, v_0) & y - y(u_0, v_0) & z - z(u_0, v_0) \\ x_u(u_0, v_0) & y_u(u_0, v_0) & z_u(u_0, v_0) \\ x_v(u_0, v_0) & y_v(u_0, v_0) & z_v(u_0, v_0) \end{vmatrix} = 0.$$

The double meaning of the term “tangential plane” is of course inconvenient, but no confusion will arise if care is taken.

According to formula (3) vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$  form a basis of the tangential plane at a point  $(u, v)$ . By a tradition borrowed from analysis the coordinates of tangent vectors relative to the basis are designated by the symbols  $du$  and  $dv$ , and the vector with those coordinates is denoted by  $d\mathbf{r}$ . Writing numerical factors at the right of the vectors we therefore get

$$(5) \quad d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv,$$

just as for numerical functions.

Now let

$$(6) \quad \mathbf{r} = \mathbf{r}_1(u_1, v_1), (u_1, v_1) \in W_1$$

be a surface equivalent to surface (1) (see Definition 3 in Lecture 24) and let

$$(7) \quad u_1 = u_1(u, v), v_1 = v_1(u, v)$$

be the corresponding diffeomorphism  $W \rightarrow W_1$ . Then

$$\mathbf{r}(u, v) = \mathbf{r}_1(u_1(u, v), v_1(u, v))$$

for any point  $(u, v) \in W$  and therefore

$$(8) \quad \begin{aligned} \mathbf{r}_u &= \frac{\partial u_1}{\partial u} (\mathbf{r}_1)_{u_1} + \frac{\partial v_1}{\partial u} (\mathbf{r}_1)_{v_1}, \\ \mathbf{r}_v &= \frac{\partial u_1}{\partial v} (\mathbf{r}_1)_{u_1} + \frac{\partial v_1}{\partial v} (\mathbf{r}_1)_{v_1}. \end{aligned}$$

It follows that the linear span of vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$  coincides with that of vectors  $(\mathbf{r}_1)_{u_1}$  and  $(\mathbf{r}_1)_{v_1}$ , i.e. the tangential plane to surface (1) at a point  $(u_1, v_1)$  coincides with the tangential plane to surface (6) at a point  $(u_1(u, v), v_1(u, v))$  (identical as a point in space with a point  $(u, v)$ ). In this sense the *tangential planes of equivalent surfaces are identical*.  $\square$

A change to equivalent surface causes in the tangential planes only a change from basis  $\mathbf{r}_u, \mathbf{r}_v$  to basis  $(\mathbf{r}_1)_{u_1}, (\mathbf{r}_1)_{v_1}$ . According to formulas (8) the corresponding transition matrix (more exactly, the matrix of inverse transition from basis  $(\mathbf{r}_1)_{u_1}, (\mathbf{r}_1)_{v_1}$  to basis  $(\mathbf{r}_u, \mathbf{r}_v)$ ) has the form

$$(9) \quad \begin{pmatrix} \frac{\partial u_1}{\partial u} & \frac{\partial u_1}{\partial v} \\ \frac{\partial v_1}{\partial u} & \frac{\partial v_1}{\partial v} \end{pmatrix},$$

i.e. is the Jacobian matrix of diffeomorphism (7).

In particular, it follows that the coordinates  $du_1, dv_1$  of tangent vectors in the basis  $(\mathbf{r}_1)_{u_1}, (\mathbf{r}_1)_{v_1}$  are related to their coordinates  $du, dv$  in the basis  $\mathbf{r}_u, \mathbf{r}_v$  by the formulas

$$(10) \quad \begin{aligned} du_1 &= \frac{\partial u_1}{\partial u} du + \frac{\partial u_1}{\partial v} dv, \\ dv_1 &= \frac{\partial v_1}{\partial u} du + \frac{\partial v_1}{\partial v} dv \end{aligned}$$

coinciding with formulas for the differentiation of formulas (7) known from analysis (this explains the choice of symbols  $du, dv$  for the coordinates of tangent vectors).

Now note that a tangential plane being a plane in Euclidean space is itself a two-dimensional Euclidean space. It

has been customary since the time of Gauss to designate the metric coefficients  $g_{11}$ ,  $g_{12}$ ,  $g_{22}$  of the basis  $\mathbf{r}_u$ ,  $\mathbf{r}_v$  of the plane by the symbols  $E$ ,  $F$ , and  $G$ . Thus by definition

$$(11) \quad E = \mathbf{r}_u^2, \quad F = \mathbf{r}_u \mathbf{r}_v, \quad G = \mathbf{r}_v^2.$$

It should be stressed that formulas (11) *define the coefficients*  $E = E(u, v)$ ,  $F = F(u, v)$ ,  $G = G(u, v)$  *as functions of*  $u$  *and*  $v$  (which is not surprising since under a change of  $u$ ,  $v$  the tangential plane is changed and so is its basis  $\mathbf{r}_u$ ,  $\mathbf{r}_v$ ).  $\square$

**Definition 2.** The quadratic form

$$E du^2 + 2F du dv + G dv^2$$

of the coordinates  $du$ ,  $dv$  of the tangent vectors relative to a basis  $\mathbf{r}_u$ ,  $\mathbf{r}_v$  is called the *first quadratic form* of surface (1) and designated by the symbol  $I$ . The value of form  $I$  on the coordinates  $du$ ,  $dv$  of a tangent vector  $d\mathbf{r}$  (designated conventionally by the symbol  $I(d\mathbf{r})$ ) is equal to the scalar square of that vector:

$$(12) \quad d\mathbf{r}^2 = I(d\mathbf{r}) = E du^2 + 2F du dv + G dv^2.$$

This means that quadratic form  $I$  is an expression in the basis  $\mathbf{r}_u$ ,  $\mathbf{r}_v$  for the quadratic functional  $d\mathbf{r} \mapsto d\mathbf{r}^2$ .

Therefore the first quadratic form  $I_1$  of the equivalent surface (6) is an expression for the same functional  $d\mathbf{r} \mapsto d\mathbf{r}^2$  but in a different basis  $(\mathbf{r}_1)_{u_1}$ ,  $(\mathbf{r}_1)_{v_1}$  and after replacement in form  $I_1$  of  $du$ ,  $dv$  with their expressions (10) form  $I$  is obtained.

For the coefficients  $E_1$ ,  $F_1$ ,  $G_1$  of form  $I_1$  this implies that they are related to the coefficients  $E$ ,  $F$ ,  $G$  of form  $I$  by the formulas

$$(13) \quad \begin{aligned} E(u, v) = & E_1(u_1, v_1) \left( \frac{\partial u_1}{\partial u} \right)^2 + 2F_1(u_1, v_1) \frac{\partial u_1}{\partial u} \frac{\partial v_1}{\partial v} + \\ & + G_1(u_1, v_1) \left( \frac{\partial v_1}{\partial u} \right)^2, \\ F(u, v) = & E_1(u_1, v_1) \frac{\partial u_1}{\partial u} \frac{\partial u_1}{\partial v} + \\ & + F_1(u_1, v_1) \left( \frac{\partial u_1}{\partial u} \frac{\partial v_1}{\partial v} + \frac{\partial u_1}{\partial v} \frac{\partial v_1}{\partial u} \right) + \\ & + G_1(u_1, v_1) \frac{\partial v_1}{\partial u} \frac{\partial v_1}{\partial v}, \end{aligned}$$

$$G(u, v) = E_1(u_1, v_1) \left( \frac{\partial u_1}{\partial v} \right)^2 + 2F_1(u_1, v_1) \frac{\partial u_1}{\partial v} \frac{\partial v_1}{\partial v} + \\ + G_1(u_1, v_1) \left( \frac{\partial v_1}{\partial v} \right)^2.$$

**Remark.** Formulas (13) can be obtained by direct calculation, substituting in formulas (11) for coefficients  $E$ ,  $F$ , and  $G$  expressions (8) for the vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ .

More loosely, under a change of coordinates on a surface its first quadratic form is linearly transformed with Jacobian matrix (9).

In other words, the *first quadratic forms of equivalent surfaces are equivalent* (at every point).  $\square$

For the tangent vector (3) of an arbitrary curve (2) on surface (1) it follows from formula (12) that for its length  $|\mathbf{r}'(t)|$  the following formula holds

$$\begin{aligned} |\mathbf{r}'(t)| &= \sqrt{I(\mathbf{r}'(t))} = \\ &= \sqrt{E(t) u'(t)^2 + 2F(t) u'(t) v'(t) + G(t) v'(t)^2}, \end{aligned}$$

where

$$\begin{aligned} E(t) &= E(u(t), v(t)), & F(t) &= F(u(t), v(t)), \\ & & G(t) &= G(u(t), v(t)). \end{aligned}$$

But according to formula (12) of Lecture 23 the length  $s$  of curve (2) between the points  $t = a$  and  $t = b$  is expressed by the formula

$$s = \int_a^b |\mathbf{r}'(t)| dt.$$

Hence

$$(14) \quad s = \int_a^b \sqrt{E(t) u'(t)^2 + 2F(t) u'(t) v'(t) + G(t) v'(t)^2} dt,$$

which may be written in the following forms, conventional but easier to remember:

$$s = \int_L \sqrt{E du^2 + 2F du dv + G dv^2},$$

$$s = \int_L \sqrt{I(d\mathbf{r})}.$$

The symbol  $L$  designates curve (2) here.

The *angle* between two space curves  $\mathbf{r} = \mathbf{r}(t)$  and  $\mathbf{r} = \mathbf{r}_1(t)$  intersecting for a given value  $t$  of the parameter is the angle  $\varphi$  between their tangent vectors  $\mathbf{r}' = \mathbf{r}'(t)$  and  $\mathbf{r}'_1 = \mathbf{r}'_1(t)$ . Hence

$$\cos \varphi = \frac{\mathbf{r}' \mathbf{r}'_1}{\sqrt{\mathbf{r}'^2} \sqrt{\mathbf{r}'_1^2}}.$$

If these curves lie on surface (1), i.e. if

$$\mathbf{r}(t) = \mathbf{r}(u(t), v(t)), \quad \mathbf{r}_1(t) = \mathbf{r}(u_1(t), v_1(t)),$$

then that formula for  $\cos \varphi$  becomes

$$(15) \quad \cos \varphi = \frac{Eu'u'_1 + F(u'v'_1 + v'u'_1) + Gv'v'_1}{\sqrt{Eu'^2 + 2Eu'v' + Gv'^2} \sqrt{Eu_1'^2 + 2Fu_1'v_1' + Gv_1'^2}}.$$

Setting

$$du = u'(t) dt, \quad dv = v'(t) dt,$$

$$\delta u = u'_1(t) dt, \quad \delta v = v'_1(t) dt$$

we may write this formula in the following conventional form

$$\cos \varphi = \frac{E du \delta u + F (du \delta v + dv \delta u) + G dv \delta v}{\sqrt{E du^2 + 2F du dv + G dv^2} \sqrt{E \delta u^2 + 2F \delta u \delta v + G \delta v^2}}$$

or in short

$$\cos \varphi = \frac{d\mathbf{r} \delta \mathbf{r}}{\sqrt{d\mathbf{r}^2} \sqrt{\delta \mathbf{r}^2}}.$$

Sometimes this formula is written in the following form which it is convenient to remember

$$\cos \varphi = \frac{I(d, \delta)}{\sqrt{I(d)} \sqrt{I(\delta)}}.$$



In particular, for the cosine of the angle between coordinate lines  $u = \text{const}$  and  $v = \text{const}$  we obtain the formula

$$\cos \varphi = \frac{F}{\sqrt{E} \sqrt{G}}.$$

Hence *coordinate lines  $u = \text{const}$  and  $v = \text{const}$  are orthogonal if and only if  $F = 0$ .*  $\square$

Now let surface (6) be an arbitrary (regular) surface not equivalent to surface (1) in general and let us be given some mapping of the support of surface (1) into the support of surface (6). In the case where surfaces (1) and (6) are injections (which we know is always true! locally, i.e. with  $W$  and  $W_1$  sufficiently small) the given mapping determines some mapping  $W \rightarrow W_1$  and conversely any mapping  $W \rightarrow W_1$  determines some mapping of the support of surface (1) into the support of surface (6). For this reason every mapping  $W \rightarrow W_1$  is called a *mapping of surface (1) into surface (6)*.

According to this definition any mapping of surface (1) into surface (6) is given by two functions

$$(16) \quad u_1 = u_1(u, v), \quad v_1 = v_1(u, v)$$

defined for  $(u, v) \in W$  and possessing the property that  $(u_1(u, v), v_1(u, v)) \in W_1$  for any point  $(u, v) \in W$ .

Mapping (16) is said to be a *diffeomorphism* of surface (1) into surface (6) if it is a diffeomorphism of an open set  $W$  onto an open set  $W_1$ .

It should be stressed that the *nonidentity functions* (16) *may give an identity mapping of supports*. It is clear that this occurs if and only if

$$\mathbf{r}(u, v) = \mathbf{r}_1(u_1(u, v), v_1(u, v))$$

for any point  $(u, v) \in W$ , i.e. (see Definition 3 in Lecture 24) *if functions (16) give the equivalence of surfaces (1) and (6).*  $\square$

On the other hand, whenever we are given some diffeomorphism (16) we can go from surface (6) to an equivalent surface

$$(17) \quad \mathbf{r} = \mathbf{r}_1(u_1(u, v), v_1(u, v))$$

and then the same mapping of supports will be given by the identity diffeomorphism  $W \rightarrow W$ . In more customary but

less precise terms this means that (with an appropriate choice of coordinates on the surfaces) *any diffeomorphism of the surfaces is a mapping defined by equating the coordinates.*  $\square$

**Definition 3.** Diffeomorphism (16) of surface (1) onto surface (6) is said to be an *isometry* if at any point  $(u, v)$  the first quadratic form of surface (17) coincides with that of surface (1), i.e. if for the coefficients of the first quadratic forms of surfaces (1) and (6) formulas (13) hold.

Surfaces are said to be *isometric* if there exists at least one isometry of one surface onto the other.

To clarify this definition consider on surface (1) an arbitrary curve  $L$ . Let as above

$$u = u(t), \quad v = v(t), \quad a \leq t \leq b,$$

be the parametric equations of that curve (as a curve on the surface). Every mapping (16) associates with the curve  $L$  a curve  $L_1$  on surface (6) with parametric equations

$$u_1 = u_1(u(t), v(t)), \quad v_1 = v_1(u(t), v(t)), \quad a \leq t \leq b.$$

It is obvious that the support of the curve  $L_1$  is the image of the support of  $L$  under the mapping of the supports of surfaces determined by mapping (16). For this reason the curve  $L_1$  is called the *image* of the curve  $L$  under mapping (16).

On the equivalent surface (17) the curve  $L_1$  is given by the same functions  $u = u(t)$ ,  $v = v(t)$  as the curve  $L$  is on surface (1). For the length  $s_1$  of the curve  $L_1$  therefore we have the formula

$$s_1 = \int_a^b \sqrt{E^* u'^2 + 2F^* u'v' + G^* v'^2} dt,$$

where  $E^*$ ,  $F^*$ ,  $G^*$  are the coefficients of the first quadratic form of surface (17). When  $E^* = E$ ,  $F^* = F$ , and  $G^* = G$ , i.e. when diffeomorphism (16) is an isometry, this formula coincides with formula (14) for the length  $s$  of the curve  $L$ . Therefore  $s_1 = s$ .

Conversely, suppose that for any curve  $L$  on surface (1) the length  $s_1$  of its image  $L_1$  on surface (6) (or, what is the same, on the equivalent surface (17)) equals the length  $s$

of the curve  $L$ . Then, in particular, this is true for the curve  $L$  given by the functions

$$u(t) = u_0 + at, \quad v(t) = v_0 + bt, \quad 0 = t = T,$$

where  $(u_0, v_0)$  is an arbitrary point in  $W$  and  $a$  and  $b$  are arbitrary numbers (and  $T$  is a number such that  $(u(t), v(t)) \in W$ , with  $0 \leq t \leq T$ ). But for these functions  $u'(t) = a$ ,  $v'(t) = b$  and therefore the equation  $s = s_1$  takes the form

$$\int_0^T \sqrt{Ea^2 + 2Fab + Gb^2} dt = \int_0^T \sqrt{E^*a^2 + 2F^*ab + G^*b^2} dt$$

from which, after differentiating with respect to  $T$  and substituting  $T = 0$ , it follows that

$$\begin{aligned} \sqrt{E(u_0, v_0)a^2 + 2F(u_0, v_0)ab + G(u_0, v_0)b^2} &= \\ &= \sqrt{E^*(u_0, v_0)a^2 + 2F^*(u_0, v_0)ab + G^*(u_0, v_0)b^2}. \end{aligned}$$

Since numbers  $a$  and  $b$  were chosen quite arbitrarily this is possible if and only if

$$\begin{aligned} E(u_0, v_0) &= E^*(u_0, v_0), \quad F(u_0, v_0) = \\ &= F^*(u_0, v_0), \quad G(u_0, v_0) = G^*(u_0, v_0), \end{aligned}$$

i.e. (since the point  $(u_0, v_0)$  was an arbitrary point in  $W$ ) if  $E = E^*$ ,  $F = F^*$ , and  $G = G^*$  in  $W$  and hence if diffeomorphism (16) is an isometry.

This proves that *a diffeomorphism of surfaces is an isometry if and only if it preserves the lengths of curves, i.e. for any curve  $L$  on surface (1) its image  $L_1$  on surface (6) has the same length.*  $\square$

On imagining a surface made of flexible but inextensible material and bending it arbitrarily we shall not change the lengths of curves on it and hence an isometric surface will result. On the basis of this intuitive idea the founders of the theory of surfaces called isometries *bendings* in the 19th century. This terminology has partly survived to this day, but now it is usual to understand bendings in a narrower sense, as isometries to be related to an identity transformation by a continuous family of isometries. All mathema-

ticians have been certain for a long time that in the local situation, i.e. in a sufficiently small neighbourhood of an arbitrary point, any isometry is a bending in that sense. Comparatively recently, however, Professor N. V. Yefimov, of Moscow University, has shown this to be false by constructing an appropriate counterexample.

Preserving lengths under isometries is a consequence of the fact that in formula (14) for the length of a curve only the coefficients of the first quadratic form  $I$  appear (besides the functions defining the curve). But formula (15) for the angle between curves also possesses this property. Therefore *angles are also preserved under isometries*.

It is convenient to give the name of the *intrinsic geometry* of a surface to the collection of all concepts and statements remaining unchanged under isometries. Thus the concepts of length and angle belong to intrinsic geometry.

It is clear that intrinsic geometry comprises every notion that, like lengths and angles, may be defined using the first quadratic form alone.

By definition, *two surfaces have the same intrinsic geometry (are isometric) when their first quadratic forms can be made identical by changing coordinates*. This test is of course quite ineffective. Therefore our immediate aim is to make it more effective. We shall deal with this in our next lecture, and now we shall consider a number of examples illustrating calculation of the first quadratic form of surfaces.

**Example 1.** A plane  $Oxy$  has in coordinates  $u = x$  and  $v = y$  a parametric equation  $\mathbf{r} = u\mathbf{i} + v\mathbf{j}$ . Therefore  $\mathbf{r}_u = \mathbf{i}$ ,  $\mathbf{r}_v = \mathbf{j}$  and hence  $E = 1$ ,  $F = 0$ ,  $G = 1$ , i.e. for the plane

$$(18) \quad I = du^2 + dv^2.$$

(A result easy to foretell without any calculations).

**Example 2.** For the circular cylinder

$$\mathbf{r} = R \cos u \cdot \mathbf{i} + R \sin u \cdot \mathbf{j} + v \cdot \mathbf{k}$$

we have  $\mathbf{r}_u = -R \sin u \cdot \mathbf{i} + R \cos u \cdot \mathbf{j}$  and  $\mathbf{r}_v = \mathbf{k}$ . Therefore

$$E = \mathbf{r}_u^2 = R^2, \quad F = \mathbf{r}_u \mathbf{r}_v = 0, \quad G = \mathbf{r}_v^2 = 1,$$

i.e. for the cylinder,

$$I = R^2 du^2 + dv^2.$$

By introducing a new coordinate  $u_1 = Ru$  (and again denoting  $u_1$  by  $u$ ) we transform this form to the form (13). Hence *a cylinder is isometric with a plane*.

Intuitively this fact is obvious: to bend a cylinder into a plane it is sufficient to cut it along its generator.

**Example 3.** For the surface of revolution

$$\mathbf{r} = x(v) \cos u \cdot \mathbf{i} + x(v) \sin u \cdot \mathbf{j} + z(v) \mathbf{k}$$

we have

$$\mathbf{r}_u = -x(v) \sin u \cdot \mathbf{i} + x(v) \cos u \cdot \mathbf{j},$$

$$\mathbf{r}_v = x'(v) \cos u \cdot \mathbf{i} + x'(v) \sin u \cdot \mathbf{j} + z'(v) \mathbf{k}.$$

Hence

$$E = x(v)^2 \sin^2 u + x(v)^2 \cos^2 u = x(v)^2,$$

$$F = -x(v) \sin u \cdot x'(v) \cos u + x(v) \cos u \cdot x'(v) \sin u = 0$$

$$G = x'(v)^2 \cos^2 u + x'(v)^2 \sin^2 u + z'(v)^2 = x'(v)^2 + z'(v)^2.$$

so that for the surface of revolution

$$I = x(v)^2 du^2 + (x'(v)^2 + z'(v)^2) dv^2.$$

It is intuitively obvious that the meridians and parallels of any surface of revolution are orthogonal. The equation  $F = 0$  could therefore be foreseen without any calculations as well.

In the case where the profile  $x = x(v)$ ,  $z = z(v)$  of a surface of revolution is referred to the natural parameter  $v = s$  (and therefore  $x'(v)^2 + z'(v)^2 = 1$ ) form  $I$  takes an especially simple form:

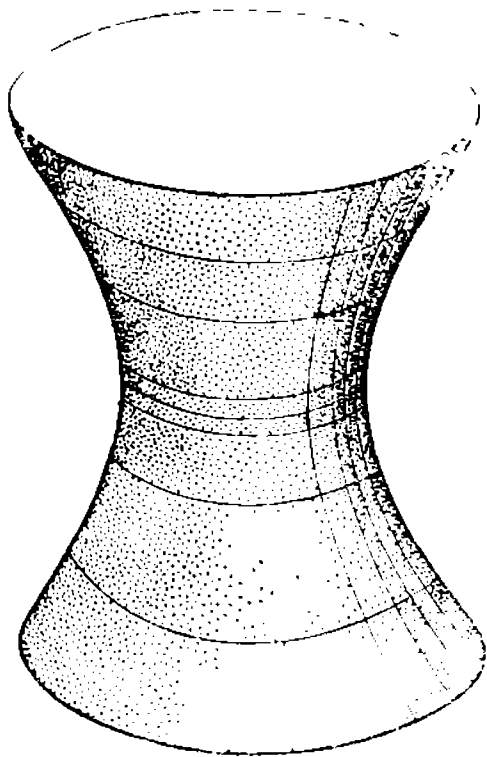
$$I = x(v)^2 du^2 + dv^2.$$

In particular we see that the first quadratic form of a sphere (of radius 1) is of the form

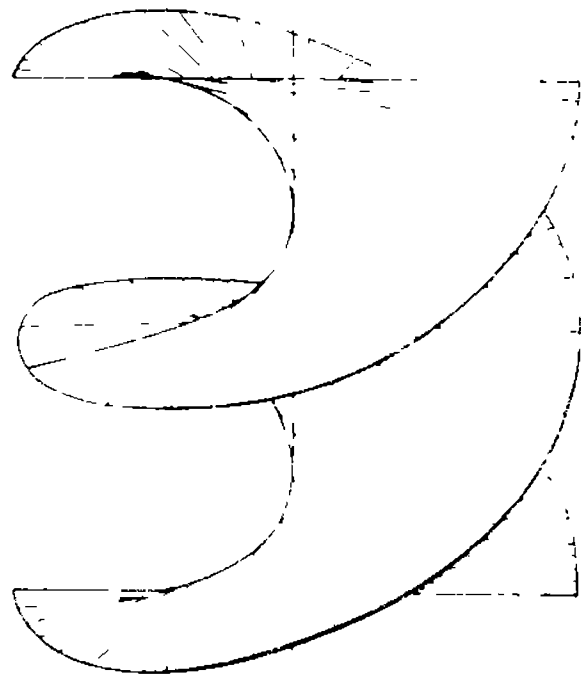
$$(19) \quad I = \cos^2 v du^2 + dv^2.$$

Cartographic experience shows that no portion of a sphere however small can be bent into a plane. This means that no transformation of coordinates can convert form (19) into form (18). But how is this to be proved? The answer will be given in our last lecture.

**Example 4.** The deflection line of a heavy homogeneous thread is called a *catenary (curve)* and a surface of revolution whose profile is a catenary is called a *catenoid*.



*A catenoid*



*A helicoid*

In mechanics (statics) it is shown that a catenary is the graph of a hyperbolic cosine. Thus for a catenoid  $x(v) = \text{ch } v$ ,  $z(v) = v$  and hence

$$x(v)^2 = \text{ch}^2 v \text{ and } x'(v)^2 + z'(v)^2 = \text{sh}^2 v + 1 = \text{ch}^2 v.$$

Thus for the catenoid

$$(20) \quad I = \text{ch}^2 v (du^2 + dv^2).$$

**Example 5.** Let a straight line perpendicular to the axis  $Oz$  rotate uniformly near it while remaining perpendicular to it and simultaneously ascending in helical motion (to a height proportional to the angle of rotation). The ruled surface swept out by that straight line is called a *helicoid*. It has the form of a helical ramp for cars to drive up.

If  $v$  is the parameter on the straight line and  $u$  is the angle of rotation, then the helicoid will have the equation

$$\mathbf{r} = v \cos u \cdot \mathbf{i} + v \sin u \cdot \mathbf{j} + u\mathbf{k}.$$

Therefore

$$\mathbf{r}_u = -v \sin u \cdot \mathbf{i} + v \cos u \cdot \mathbf{j} + \mathbf{k},$$

$$\mathbf{r}_v = \cos u \cdot \mathbf{i} + \sin u \cdot \mathbf{j},$$

and hence

$$E = 1 + v^2, \quad F = 0, \quad G = 1.$$

Thus for a helicoid

$$I = (1 + v^2) du^2 + dv^2.$$

Let us transform this form by introducing new coordinates  $u_1, v_1$  related to the coordinates  $u, v$  by the formulas

$$u = u_1, \quad v = \operatorname{sh} v_1.$$

Then

$$1 + v^2 = 1 + \operatorname{sh}^2 v_1 = \operatorname{ch}^2 v_1,$$

$$du = du_1, \quad dv = \operatorname{ch} v_1 dv_1,$$

and therefore (we drop the indices in the new coordinates)

$$I = \operatorname{ch}^2 v (du^2 + dv^2),$$

which coincides with form (20).

This proves that the *catenoid and the helicoid are isometric* (only locally of course), there existing an isometry transforming meridians of the catenoid into rectilinear generators of the helicoid.

An astonishing result!

**Example 6.** For an arbitrary ruled surface

$$(21) \quad \mathbf{r} = \boldsymbol{\rho}(u) + v\mathbf{a}(u),$$

where (see the preceding lecture)  $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$  is a regular curve referred to the natural parameter and  $\mathbf{a}(u)$  is a vector function such that  $|\mathbf{a}(u)| = 1$  for all  $u$ , denoting differentiation with respect to  $u$  with a dot, we have

$$\mathbf{r}_u = \dot{\boldsymbol{\rho}} + v\dot{\mathbf{a}}, \quad \mathbf{r}_v = \mathbf{a}.$$

Since  $\dot{\rho}^2 = 1$ , and  $\dot{a}a = 0$  and  $a^2 = 1$ , we have

$$E = 1 + 2v\ddot{\rho}a + v^2\ddot{a}^2, \quad F = \dot{\rho}a, \quad G = 1.$$

If in particular  $a = \dot{\rho}$  (a surface of tangents), then  $\dot{\rho}a = a^2 = 1$  (i.e.  $F = 1$ ) and  $\ddot{\rho}a = 0$  and  $\ddot{a}^2 = k^2$ , where  $k$  is the curvature of the curve  $\rho = \rho(u)$  (i.e.  $E = (1 + k^2v^2)$ ). Thus for a surface of tangents

$$(22) \quad I = (1 + k^2v^2) du^2 + 2du dv + dv^2.$$

But if  $a(u)$  is the binormal vector of the curve  $\rho = \rho(u)$ , then  $\dot{\rho}a = 0$ ,  $\ddot{\rho}a = 0$  and  $\ddot{a}^2 = \kappa^2$ , where  $\kappa$  is the torsion of the curve  $\rho = \rho(u)$ . Hence for a surface of binormals

$$I = (1 + \kappa^2v^2) du^2 + dv^2.$$

We thus see that the first quadratic form of a surface of tangents depends only on the curvature of a given curve and that the first quadratic form of a surface of binormals depends only on the torsion of the surface.

For surfaces of tangents this implies that *every surface of tangents is isometric with a plane* (locally). Indeed, consider a plane curve with the same curvature  $k = k(u)$  (such a curve exists by virtue of Theorem 1 of Lecture 24). The first quadratic form of the surface of tangents of that curve is the same form (22). But, on the other hand, it is clear that a surface of tangents of a plane curve is (locally) a plane. There exists therefore a change of coordinates transforming the first quadratic form  $dx^2 + dy^2$  of the plane into form (22). (This change of coordinates has the form

$$x = x(u) + x'(u)v, \quad y = y(u) + y'(u)v,$$

where  $x(u)$  and  $y(u)$  are functions such that  $x'(u)^2 + y'(u)^2 = 1$  and  $x''(u)^2 + y''(u)^2 = k(u)^2$ .)  $\square$

This isometry can be carried out by continuous bending, gradually deforming the curve  $\rho = \rho(u)$  into a plane curve.

For this reason surfaces of tangents are called *developable surfaces* (or *developables*) (development into a plane is meant).



If  $\mathbf{a}(u) = \rho(u)$ , surface (21) is a cone with vertex at the origin (and the curve  $\rho = \rho(u)$  is the intersection of the cone with a unit sphere  $|\rho| = 1$ ). In this case we have

$$\dot{\rho}\dot{\mathbf{a}} = \rho^2 = 1, \quad \mathbf{a}^2 = 1, \quad \dot{\rho}\mathbf{a} = 0,$$

so that form  $I$  becomes

$$I = (1 + v)^2 du^2 + dv^2.$$

Here the change of coordinates  $(u, v) \mapsto (u, 1 + v)$  suggests itself, converting the last form into a slightly simpler form

$$(23) \quad I = v^2 du^2 + dv^2.$$

Now let us introduce new coordinates

$$x = v \cos u, \quad y = v \sin u.$$

Then

$$dx = -v \sin u \, du + \cos u \, dv,$$

$$dy = v \cos u \, du + \sin u \, dv,$$

and hence

$$dx^2 + dy^2 = v^2 du^2 + dv^2.$$

This proves that *any cone is isometric with a plane*. For this reason cones are also reckoned among developables.

Note that form (23) is nothing but the first quadratic form of a plane referred to polar coordinates  $r = v$  and  $\varphi = u$ .

Finally, if the vector  $\mathbf{a}(u)$  is constant (and therefore  $\dot{\mathbf{a}} = 0$ ), surface (21) is a cylinder. We may consider without loss of generality that the directrix  $\rho = \rho(u)$  of the cylinder is a plane curve whose plane is orthogonal to the vector  $\mathbf{a}$  (and hence  $\rho\mathbf{a} = 0$  and  $\dot{\rho}\mathbf{a} = 0$ ). Therefore, as with the circular cylinder (Example 2),

$$I = du^2 + dv^2.$$

For this reason all cylinders are also reckoned among developables.

In the next lecture we shall show that *among ruled surfaces only developables* (i.e. cylinders, cones, and surfaces of tangents) *are isometric with a plane*. Moreover, it turns out that developables exhaust *all* the surfaces isometric with a plane. We shall leave this fact without proof.

# Lecture 26

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*The tangential plane and the normal vector. The curvature of a normal section. The second quadratic form of a surface. The indicatrix of Dupin. Principal curvatures. The second quadratic form of a graph. Ruled surfaces of zero curvature. Surfaces of revolution*

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We proceed to consider an arbitrary regular surface

$$(1) \quad \mathbf{r} = \mathbf{r}(u, v), \quad (u, v) \in W$$

in a three-dimensional Euclidean space  $\mathcal{E}$ .

Recall (see the preceding lecture) that the tangential plane at a point  $(u, v)$  of surface (1) is a plane in space passing through a point with radius vector  $\mathbf{r}(u, v)$  and having the direction bivector  $\mathbf{r}_u \wedge \mathbf{r}_v$ . If the space  $\mathcal{E}$  is oriented, then for any point  $(u, v)$  of the surface a unit vector  $\mathbf{n} = \mathbf{n}(u, v)$  is defined perpendicular to the tangential plane and constituting, together with vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ , a positively oriented basis

$$(2) \quad \mathbf{r}_u, \mathbf{r}_v, \mathbf{n}$$

of the space  $\mathcal{E}$  (more precisely, of its associated vector space  $\mathcal{V}$ ). That basis is called the *normal vector* to surface (1) at a point  $(u, v)$ . Basis (2) is called the *moving basis* of the surface at the point  $(u, v)$ .

It should be stressed that the moving basis is not orthonormal in general.

The vector  $\mathbf{n}$  is of course collinear with the vector  $\mathbf{r}_u \times \mathbf{r}_v$ . Hence

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}.$$

**Lemma 1.** *For any two vectors  $\mathbf{a}$ ,  $\mathbf{b}$  of a three-dimensional oriented Euclidean vector space  $\mathcal{V}$  we have*

$$|\mathbf{a} \times \mathbf{b}|^2 = \begin{vmatrix} \mathbf{a}^2 & \mathbf{a}\mathbf{b} \\ \mathbf{a}\mathbf{b} & \mathbf{b}^2 \end{vmatrix}.$$

*Proof.* Let  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$  be a positively oriented orthonormal basis of a vector space  $\mathcal{V}$  such that

$$\begin{aligned} \mathbf{a} &= a\mathbf{i}, \\ \mathbf{b} &= b'\mathbf{i} + b\mathbf{j}. \end{aligned}$$

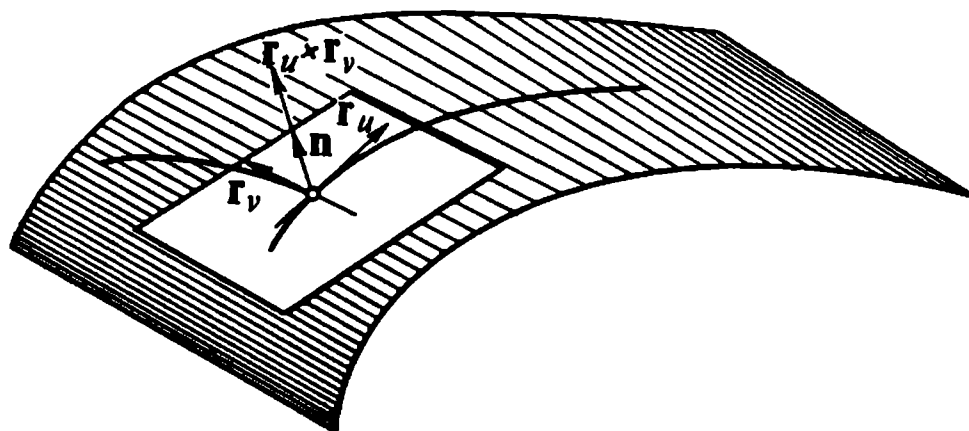
Then  $\mathbf{a} \times \mathbf{b} = ab\mathbf{k}$  and

$$\mathbf{a}^2 = a^2, \quad \mathbf{a}\mathbf{b} = ab', \quad \mathbf{b}^2 = b'^2 + b^2.$$

Therefore  $|\mathbf{a} \times \mathbf{b}| = a^2b^2$  and

$$\begin{vmatrix} \mathbf{a}^2 & \mathbf{a}\mathbf{b} \\ \mathbf{a}\mathbf{b} & \mathbf{b}^2 \end{vmatrix} = \begin{vmatrix} a^2 & ab' \\ ab' & b'^2 + b^2 \end{vmatrix} = a^2(b'^2 + b^2) - (ab')^2 = a^2b^2. \quad \square$$

**Remark.** In any Euclidean space a theory of volumes can be developed quite similar to an elementary theory of areas



and volumes in three-dimensional space. Then Lemma 1 will turn out to be a special case of the general proposition stating that for any vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$  of an arbitrary Euclidean vector space  $\mathcal{V}$  the square of the  $m$ -dimensional vol-

ume of a parallelepiped constructed on those vectors is equal to the determinant

$$\begin{vmatrix} \mathbf{a}_1^2 & \mathbf{a}_1\mathbf{a}_2 & \dots & \mathbf{a}_1\mathbf{a}_m \\ \mathbf{a}_2\mathbf{a}_1 & \mathbf{a}_2^2 & \dots & \mathbf{a}_2\mathbf{a}_m \\ \dots & \dots & \dots & \dots \\ \mathbf{a}_m\mathbf{a}_1 & \mathbf{a}_m\mathbf{a}_2 & \dots & \mathbf{a}_m^2 \end{vmatrix}.$$

This is called the *Gramian* of vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$ . It is zero if and only if these vectors are linearly dependent. If  $m = \dim \mathcal{V}$  and the vectors  $\mathbf{a}_1, \dots, \mathbf{a}_m$  are linearly independent (constitute a basis), the elements of the Gramian are nothing but the metric coefficients of that basis.

On applying Lemma 1 to the vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$  we at once see that

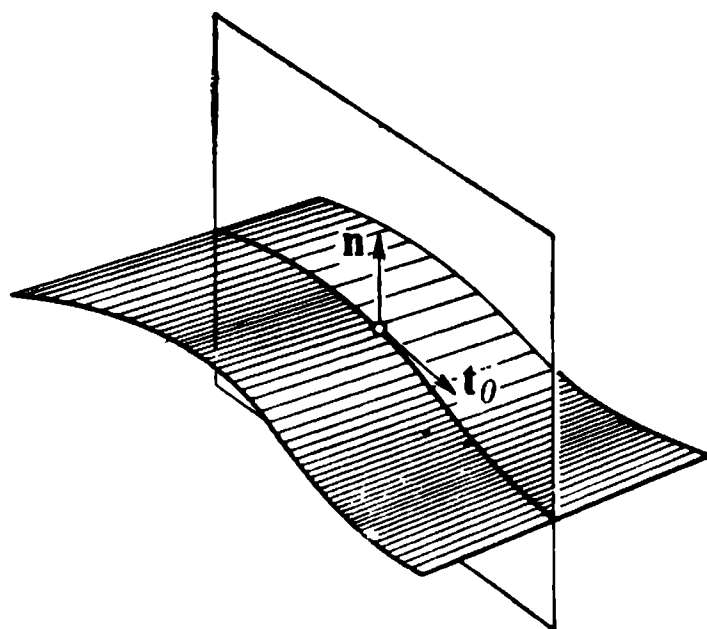
$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = \begin{vmatrix} \mathbf{r}_u^2 & \mathbf{r}_u\mathbf{r}_v \\ \mathbf{r}_v\mathbf{r}_u & \mathbf{r}_v^2 \end{vmatrix} = \begin{vmatrix} E & F \\ F & G \end{vmatrix} = EG - F^2,$$

and hence that

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{\sqrt{EG - F^2}}$$

It is by this formula that the vector  $\mathbf{n}$  is usually computed.

Let  $\mathbf{t}_0$  be an arbitrary unit vector which is the tangent vector of surface (1) at a point  $(u_0, v_0)$ . Consider a plane passing through a point with radius vector  $\mathbf{r}(u_0, v_0)$  and having a direction bivector  $\mathbf{t}_0 \wedge \mathbf{n}_0$ , where  $\mathbf{n}_0 = \mathbf{n}(u_0, v_0)$ . It is intuitively obvious that the plane intersects the surface in some curve having at the point  $(u_0, v_0)$  the tangent vector  $\mathbf{t}_0$  (and hence regular). This curve is called the *normal section* of surface (1) determined by the tangent vector  $\mathbf{t}_0$ .



*Normal section*

Let rectangular coordinates  $x, y, z$  be chosen in a space  $\mathcal{E}$  so that surface (1) in the neighbourhood of a point  $(u_0, v_0)$

is the graph of a smooth function  $z = z(x, y)$ , with  $\mathbf{n}_0$  being the coordinate unit vector  $\mathbf{k}$ . Then, if  $\mathbf{t}_0 = a\mathbf{i} + b\mathbf{j}$ , the normal section determined by the vector  $\mathbf{t}_0$  will obviously have (as a curve on the surface) equations

$$u = u_0 + at, \quad v = v_0 + bt$$

(in space this curve would have equations  $x = u_0 + at$ ,  $y = v_0 + bt$ ,  $z = z(u_0 + at, v_0 + bt)$ ).

This not only provides a method of writing the equations of a normal section, but also allows its formal definition (not based on intuition) as a curve on the surface with equations  $u = u_0 + at$ ,  $v = v_0 + bt$  (provided of course surface (1) is represented as the graph of the smooth function  $z = z(x, y)$ ). It is certainly required here to verify the correctness of this definition, which is in principle not hard to do. We shall not deal with this, however, since the notion of normal section will play in our discussion only an auxiliary and mainly heuristic role.

Let  $u = u(s)$ ,  $v = v(s)$  be the equations (on the surface) of the normal section of surface (1) at a point  $(u_0, v_0)$ , determined by the tangent vector  $\mathbf{t}_0$ . Suppose that  $s$  is the natural parameter of a space curve  $\mathbf{r} = \mathbf{r}(s)$ , where  $\mathbf{r}(s) = \mathbf{r}(u(s), v(s))$ , with  $u(0) = u_0$ ,  $v(0) = v_0$ . Then for the tangent vector  $\mathbf{t} = \mathbf{t}(s)$  of the normal section we have

$$\mathbf{t} = \dot{\mathbf{r}} = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v},$$

with  $\mathbf{t}(0) = \mathbf{t}_0$ . Hence

$$\begin{aligned} \dot{\mathbf{t}} &= \dot{\mathbf{r}}_u \dot{u} + \mathbf{r}_{uu} \ddot{u} + \dot{\mathbf{r}}_v \dot{v} + \mathbf{r}_{vv} \ddot{v} = \\ &= (\mathbf{r}_{uu} \dot{u} + \mathbf{r}_{uv} \dot{v}) \dot{u} + (\mathbf{r}_{vu} \dot{u} + \mathbf{r}_{vv} \dot{v}) \dot{v} + \mathbf{r}_{uu} \ddot{u} + \mathbf{r}_{vv} \ddot{v} = \\ &= \mathbf{r}_{uu} (\dot{u})^2 + 2\mathbf{r}_{uv} (\dot{u}\dot{v}) + \mathbf{r}_{vv} (\dot{v})^2 + \mathbf{r}_{uu} \ddot{u} + \mathbf{r}_{vv} \ddot{v}. \end{aligned}$$

Putting here  $s = 0$  and multiplying by  $\mathbf{n}_0$  we get

$$\begin{aligned} (3) \quad \dot{\mathbf{t}}(0) \mathbf{n}_0 &= ((\mathbf{r}_{uu})_0 \mathbf{n}_0) \dot{u}(0)^2 + \\ &\quad + 2((\mathbf{r}_{uv})_0 \mathbf{n}_0) \dot{u}(0) \dot{v}(0) + ((\mathbf{r}_{vv})_0 \mathbf{n}_0) \dot{v}(0)^2, \end{aligned}$$

for  $(\mathbf{r}_u)_0 \mathbf{n}_0 = 0$  and  $(\mathbf{r}_v)_0 \mathbf{n}_0 = 0$ .

Now note that by definition a normal section is a plane curve. In the plane of the curve the vectors  $\mathbf{t}_0, \mathbf{n}_0$  determine some orientation and with respect to that orientation the normal section will have at each of its points relative curvature  $k_{rel}$  (see Lecture 22). At the point  $s = 0$  the curvature is obviously equal to the scalar product  $\dot{\mathbf{t}}(0) \mathbf{n}_0$  we have just computed and is therefore expressed by formula (3).

To simplify the formulas we shall now drop the index zero everywhere, i.e. denote the vector  $\mathbf{t}_0$  by  $\mathbf{t}$ , the point  $(u_0, v_0)$  by  $(u, v)$  and so on. The relative curvature (at the point  $s = 0$ ) of the normal section at a point  $(u, v)$ , determined by the vector  $\mathbf{t}$ , will be denoted by  $k(\mathbf{t})$ . Besides, we set

$$\begin{aligned} L &= \mathbf{r}_{uu} \mathbf{n} = -\mathbf{r}_u \mathbf{n}_u, \\ M &= \mathbf{r}_{uv} \mathbf{n} = -\mathbf{r}_u \mathbf{n}_v = -\mathbf{r}_v \mathbf{n}_u, \\ N &= \mathbf{r}_{vv} \mathbf{n} = -\mathbf{r}_v \mathbf{n}_v \end{aligned}$$

(since  $\mathbf{r}_u \mathbf{n} = 0$ , we have  $\mathbf{r}_{uu} \mathbf{n} + \mathbf{r}_u \mathbf{n}_u = 0$  and  $\mathbf{r}_{uv} \mathbf{n} + \mathbf{r}_u \mathbf{n}_v = 0$  and since  $\mathbf{r}_v \mathbf{n} = 0$ , we have  $\mathbf{r}_{vu} \mathbf{n} + \mathbf{r}_v \mathbf{n}_u = 0$  and  $\mathbf{r}_{vv} \mathbf{n} + \mathbf{r}_v \mathbf{n}_v = 0$ ). In this notation formula (3) takes the form

$$(5) \quad k(\mathbf{t}) = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2,$$

where  $\dot{u}$  and  $\dot{v}$  are the coordinates of the vector  $\mathbf{t}$  in the basis  $\mathbf{r}_u, \mathbf{r}_v$ :

$$\mathbf{t} = \mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}.$$

Formula (5) may now be taken as a formal definition of a function  $\mathbf{t} \mapsto k(\mathbf{t})$ , and all said above regarded as merely an informal motivation of the definition.

It is convenient to extend the function  $\mathbf{t} \mapsto k(\mathbf{t})$  constructed now to include all possible nonzero tangent vectors  $d\mathbf{r} = \mathbf{r}_u du + \mathbf{r}_v dv$  assuming by definition that

$$k(d\mathbf{r}) = k\left(\frac{d\mathbf{r}}{ds}\right)$$

(recall that  $ds = | d\mathbf{r} |$ ; see above). Since the coordinates of the unit vector  $\frac{d\mathbf{r}}{ds}$  are the numbers  $\frac{du}{ds}$  and  $\frac{dv}{ds}$ , we have by formula (5)

$$\begin{aligned} k(d\mathbf{r}) &= L \left( \frac{du}{ds} \right)^2 + 2M \frac{du}{ds} \frac{dv}{ds} + N \left( \frac{dv}{ds} \right)^2 = \\ &= \frac{L du^2 + 2M du dv + N dv^2}{ds^2} . \end{aligned}$$

Since

$$ds^2 = E du^2 + 2F du dv + G dv^2,$$

it follows that

$$(6) \qquad k(d\mathbf{r}) = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2} .$$

**Definition 1.** The quadratic form

$$L du^2 + 2M du dv + N dv^2$$

is called the *second quadratic form* of surface (1). It is designated by the symbol  $II$ .

Introducing the vector

$$(7) \qquad d\mathbf{n} = \mathbf{n}_u du + \mathbf{n}_v dv$$

form  $II$  can be identified (by virtue of (4)) with the scalar product  $-d\mathbf{r} d\mathbf{n}$ .

Formula (6) can now be written in the following form convenient to remember:

$$k = \frac{II}{I}$$

or, using vector (7), in the form

$$k = - \frac{d\mathbf{r} d\mathbf{n}}{d\mathbf{r}^2}$$

In the literature symbols  $D, D', D''$  are also used to designate the coefficients  $L, M, N$  of form  $II$ .

To visualize the function  $\mathbf{t} \mapsto k(\mathbf{t})$  the French mathematician Dupin suggested that on the tangent plane the curve (now called the *indicatrix of Dupin*) should be considered that results if for any unit vector  $\mathbf{t}$  a segment of length

$|k(t)|^{-1/2}$  is marked off from the point of tangency (taken as the origin  $O$  on the tangent plane) in the direction of that vector. Denote by  $x$  and  $y$  the coordinates (in the coordinate system  $Or_u r_v$ ) of the terminal point of the segment; then its length is expressible (in clear notation) by formula

$$|x\mathbf{r}_u + y\mathbf{r}_v| = \sqrt{I(x, y)}.$$

Since the curvature  $k(t)$  can be expressed by formula (6), which in the present notation has the form

$$k(t) = \frac{II(x, y)}{I(x, y)},$$

we obtain for the indicatrix of Dupin the equation

$$\sqrt{I(x, y)} = \sqrt{\frac{I(x, y)}{II(x, y)}},$$

i.e. the equation

$$|II(x, y)| = 1.$$

This proves that *the indicatrix of Dupin is a curve with equation*

$$|Lx^2 + 2Mxy + Ny^2| = 1.$$

When  $LN - M^2 > 0$  the curve (more precisely, the set of its real points which is our only concern) is an ellipse with equation

$$(8) \quad Lx^2 + 2Mxy + Ny^2 = \varepsilon,$$

where  $\varepsilon = +1$  if  $L > 0$  and  $\varepsilon = -1$  if  $L < 0$ . Accordingly a point of surface (1) at which  $LN - M^2 > 0$  is called *elliptical*.

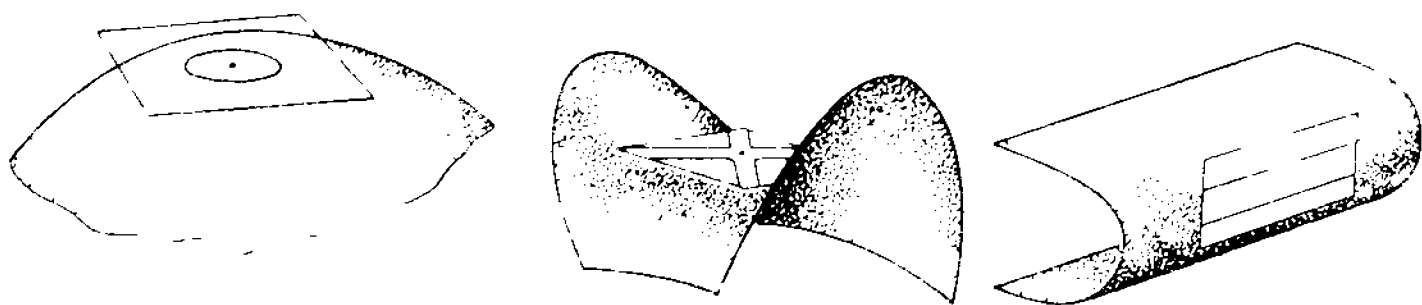
At an elliptical point all curvatures  $k(t)$  have the same sign (coinciding with that of  $L$ ). Among them, there is one maximum  $k_1$  and one minimum  $k_2$  (unless they all coincide, i.e. unless the indicatrix of Dupin is a circle) corresponding to the directions of the minor and major axes of ellipse (8).

When  $LN - M^2 < 0$  the indicatrix of Dupin consists of two hyperbolas

$$(9) \quad Lx^2 + 2Mxy + Ny^2 = \pm 1$$



with common asymptotes and therefore a point of surface (1) at which  $LN - M^2 < 0$  is called *hyperbolic*. In the direction of the real axis of one of the hyperbolas (9) the curvature  $k(t)$  attains its maximum value  $k_1 > 0$ . As the vector  $t$  is rotated the curvature first decreases to zero, when the vector  $t$  assumes asymptotic direction, and then, continuing to decrease, attains its minimum value  $k_2 < 0$ , when the direction of the vector  $t$  coincides with that of the real axis



At an elliptical point      At a hyperbolic point      At a parabolic point  
The indicatrix of Dupin

of the other hyperbola (i.e. with the direction of the imaginary axis of the first hyperbola).

When  $LN - M^2 = 0$  a point of surface (1) is called *parabolic*. At such a point the indicatrix of Dupin has the equation

$$(10) \quad (V|L|x + V|N|y)^2 = 1$$

and therefore is a pair of parallel lines (provided  $L \neq 0$  or  $N \neq 0$ ). In the direction of these lines the curvature  $k(t)$  is equal to zero, in the perpendicular direction it reaches its maximum (in magnitude) maintaining throughout the same sign. But if  $L = 0$ ,  $N = 0$  (and therefore  $M = 0$ ), the curvature  $k(t)$  is identically as a function of  $t$  equal to zero (and the indicatrix of Dupin is not defined).

Note that at elliptical and parabolic points the indicatrix of Dupin is a second degree curve, and at hyperbolic points it is a quartic curve.

In each of the three cases the function  $k(t)$  *twice* attains its maximum  $k_1$  and its minimum  $k_2$  (unless it is identically equal to zero).

**Definition 2.** Numbers  $k_1$  and  $k_2$  are called the *principal curvatures* of surface (1) at the point under consideration. Their product

$$K = k_1 k_2$$

is called the *total* (or *Gaussian*) *curvature* and their half-sum

$$H = \frac{k_1 + k_2}{2}$$

is termed *mean curvature*.

According to the above said,  $K > 0$  at an elliptical point,  $K < 0$  at a hyperbolic point, and  $K = 0$  at a parabolic point.

To find principal curvatures one could seek the principal directions of the second degree curves (8) and (9) (there is no problem with curve (10)) and then find their canonical equations. Unfortunately, this method involves lengthy computations because the coordinates  $x$  and  $y$  are not rectangular. Therefore we shall proceed in a different way, applying directly to the basic formula (6).

According to this formula curvature  $k_2$  is the smallest value of the function

$$\frac{II(x, y)}{I(x, y)} = \frac{Lx^2 + 2Mxy + Ny^2}{Ex^2 + 2Fxy + Gy^2}$$

of two variables  $x$  and  $y$ , with  $(x, y) \neq (0, 0)$ . Hence

$$\frac{II(x, y)}{I(x, y)} \geq k_2$$

for all  $(x, y) \neq (0, 0)$ , equality holding at least at one point  $(x, y)$ . Since  $I(x, y) > 0$  when  $(x, y) \neq (0, 0)$ , this inequality is the same as the inequality

$$II(x, y) - k_2 I(x, y) \geq 0$$

implying that the quadratic form  $II - k_2 I$  with matrix

$$\begin{pmatrix} L - k_2 E & M - k_2 F \\ M - k_2 F & N - k_2 G \end{pmatrix}$$

is nonnegative at all points  $(x, y) \neq (0, 0)$  and zero at least at one of them.

Similarly, the number  $k_1$  is characterized by the fact that the quadratic form  $II - k_1 I$  is everywhere nonpositive and zero at least at one point  $(x, y) \neq (0, 0)$ .

But it is easy to see (directly or on the basis of the general theory of quadratic forms over the field  $\mathbb{R}$ ; see Lecture 12) that *a quadratic form in two variables is everywhere nonpositive or nonnegative and zero at least at one point  $(x, y) \neq (0, 0)$  if and only if its rank is less than two, i.e. if the determinant of its matrix is zero.*  $\square$

This proves that the principal curvatures  $k_1, k_2$  are the roots of the equation

$$\begin{vmatrix} L - kE & M - kF \\ M - kF & N - kG \end{vmatrix} = 0,$$

i.e. of the equation

$$(EG - F^2) k^2 - (EN + GL - 2FM) k + (LN - M^2) = 0.$$

In particular it follows (by virtue of Viète's formulas) that

$$K = \frac{LN - M^2}{EG - F^2}, \quad H = \frac{1}{2} \frac{EN + GL - 2FM}{EG - F^2}.$$

The first of these formulas will find an important application in our next lecture.

Suppose that coordinates  $x, y, z$  in a space  $\mathcal{E}$  have been chosen so that the surface under consideration is the graph of a function  $z = z(x, y)$ , with  $z(0, 0) = 0$ , and the normal vector at the point  $(0, 0)$  is the unit vector  $\mathbf{k}$  of the axis  $Oz$ . It is easy to see that the last assumption is the same as the assumption that  $\left(\frac{\partial z}{\partial x}\right)_0 = 0$  and  $\left(\frac{\partial z}{\partial y}\right)_0 = 0$ . Hence expansion of the function  $z(x, y)$  into a Taylor series begins with quadratic terms:

$$z = rx^2 + 2sxy + ty^2 + \dots,$$

where

$$r = \left(\frac{\partial^2 z}{\partial^2 x}\right)_0, \quad s = \left(\frac{\partial^2 z}{\partial x \partial y}\right)_0, \quad t = \left(\frac{\partial^2 z}{\partial y^2}\right)_0.$$

Since in this case  $\mathbf{r} = u\mathbf{i} + v\mathbf{j} + z(u, v)\mathbf{k}$ , we have  $\mathbf{r}_u = \mathbf{i} + z_u\mathbf{k}$ ,  $\mathbf{r}_v = \mathbf{j} + z_v\mathbf{k}$  and  $\mathbf{r}_{uu} = z_{uu}\mathbf{k}$ ,  $\mathbf{r}_{uv} = z_{uv}\mathbf{k}$ ,  $\mathbf{r}_{vv} = z_{vv}\mathbf{k}$ . Hence at the point  $(0, 0)$  we have  $L = r$ ,  $M = s$ ,  $N = t$ , i.e. in the case under consideration the *second quadratic form coincides with the sum  $z_2(x, y)$  of quadratic terms in the Taylor series of the function  $z(x, y)$* .  $\square$

Since near the point  $(0, 0)$  the surface  $z = z(x, y)$  differs but little from the surface  $z = z_2(x, y)$  and since for  $rt - s^2 > 0$  the latter surface is an elliptical paraboloid and for  $rt - s^2 < 0$  it is a hyperbolic paraboloid, this proves that *an arbitrary surface differs but little from the elliptical paraboloid near an elliptical point and from the hyperbolic paraboloid near a hyperbolic point*.  $\square$

This gives a quite satisfactory idea of the behaviour of the surface near nonparabolic points.

As to the behaviour of the surface near a parabolic point nothing definite can be said about it; it may be very complex in general.

For the ruled surface

$$(11) \quad \dot{\mathbf{r}} = \dot{\boldsymbol{\rho}}(u) + v\dot{\mathbf{a}}(u),$$

as we already know,

$$E = 1 + 2v\dot{\boldsymbol{\rho}}\ddot{\mathbf{a}} + v^2\dot{\mathbf{a}}^2, \quad F = \dot{\boldsymbol{\rho}}\dot{\mathbf{a}}, \quad G = 1$$

(we as ever assume that the parameter  $u$  on the curve  $\boldsymbol{\rho} = \boldsymbol{\rho}(u)$  is natural and the vector  $\mathbf{a}(u)$  is a unit vector. Further

$$\mathbf{r}_u = \dot{\boldsymbol{\rho}} + v\dot{\mathbf{a}}, \quad \mathbf{r}_v = \dot{\mathbf{a}},$$

$$\mathbf{r}_u \times \mathbf{r}_v = \dot{\boldsymbol{\rho}} \times \dot{\mathbf{a}} + v(\dot{\mathbf{a}} \times \dot{\mathbf{a}}),$$

$$\mathbf{n} = \frac{\dot{\boldsymbol{\rho}} \times \dot{\mathbf{a}} + v(\dot{\mathbf{a}} \times \dot{\mathbf{a}})}{\sqrt{EG - F^2}},$$

$$\mathbf{r}_{uu} = \ddot{\boldsymbol{\rho}} + v\ddot{\mathbf{a}}, \quad \mathbf{r}_{uv} = \dot{\mathbf{a}}, \quad \mathbf{r}_{vv} = 0,$$

$$L = \frac{(\ddot{\boldsymbol{\rho}} + v\ddot{\mathbf{a}})(\dot{\boldsymbol{\rho}} \times \dot{\mathbf{a}} + v(\dot{\mathbf{a}} \times \dot{\mathbf{a}}))}{\sqrt{EG - F^2}}, \quad M = \frac{\dot{\boldsymbol{\rho}}\ddot{\mathbf{a}}}{\sqrt{EG - F^2}}, \quad N = 0,$$

$$LN - M^2 = -\frac{(\dot{\boldsymbol{\rho}}\ddot{\mathbf{a}})^2}{EG - F^2},$$

and therefore

$$K = -\frac{(\dot{\rho} \mathbf{a} \dot{\mathbf{a}})^2}{(EG - F^2)^2} \leq 0.$$

Thus *the total curvature of an arbitrary ruled surface is non-positive, i.e. a ruled surface has no elliptical points.*  $\square$

When the surface is a cylinder ( $\dot{\mathbf{a}} = 0$ ), a cone ( $\mathbf{a} = \dot{\rho}$  and therefore  $\dot{\mathbf{a}} = \dot{\rho}$ ) or a surface of tangents ( $\mathbf{a} = \dot{\rho}$ ), the formula obtained yields  $K = 0$ . Thus *the total curvature of every developable is equal to zero.*

Conversely, if  $K = 0$ , then  $\dot{\rho} \mathbf{a} \dot{\mathbf{a}} = 0$ , i.e. the vectors  $\dot{\rho}$ ,  $\mathbf{a}$ ,  $\dot{\mathbf{a}}$  are coplanar. If the vector  $\dot{\mathbf{a}}(u)$  is not identically zero, i.e. if surface (11) is not a cylinder, then, passing, if necessary, to a smaller neighbourhood, we may assume that  $\dot{\mathbf{a}}(u) \neq 0$  for all  $u$ . The vectors  $\mathbf{a}$  and  $\dot{\mathbf{a}}$  are therefore linearly independent (they are nonzero and orthogonal) and hence the vector  $\dot{\rho}$  is linearly expressible in terms of them:

$$\dot{\rho} = \lambda \mathbf{a} + \mu \dot{\mathbf{a}},$$

where  $\lambda = \lambda(u)$ ,  $\mu = \mu(u)$  are some functions of  $u$ .

Let

$$u_1 = u, \quad v_1 = v + \mu(u).$$

Since the Jacobian of this transformation is equal to 1, the numbers  $u_1$  and  $v_1$  are also, after possibly passing to a smaller neighbourhood, coordinates on surface (11), i.e., to be more exact, they determine an equivalent surface. The equation of that surface is of the form

$$\mathbf{r} = \rho_1(u_1) + v_1 \mathbf{a}(u_1),$$

where

$$\rho_1(u) = \rho(u) - \mu(u) \mathbf{a}(u),$$

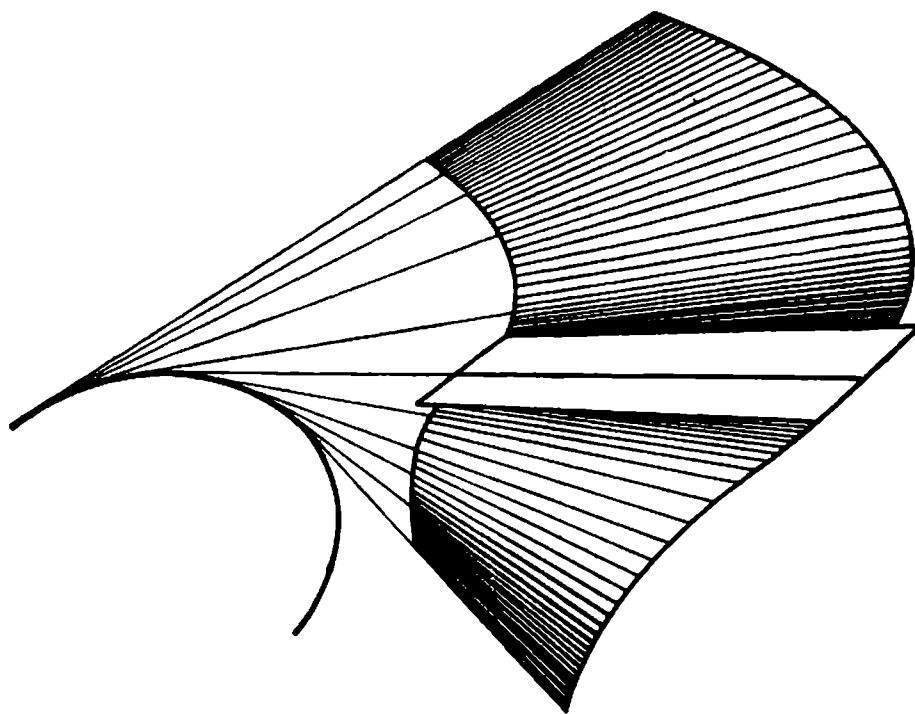
and so

$$\dot{\rho}_1 = \dot{\rho} - \dot{\mu} \mathbf{a} - \mu \dot{\mathbf{a}} = (\lambda - \dot{\mu}) \mathbf{a}.$$

If  $\dot{\rho}_1 = 0$  identically (i.e.  $\lambda = \dot{\mu}$ ), then the equation of surface (11) is of the form

$$\mathbf{r} = \text{const} + v_1 \mathbf{a}(u_1)$$

and therefore that surface is a cone. Otherwise we may assume, diminishing if necessary the neighbourhood, that  $\dot{\rho}_1(u) \neq 0$  for all  $u$ . Passing then to the natural parameter



*A developable surface of tangents*

(and changing if necessary the sign of  $v_1$ ) we see that  $\rho_1 = \mathbf{a}$ , i.e. that the surface under consideration is a surface of tangents.

Thus we have proved the following proposition:

**Proposition 1.** *A ruled surface has zero total curvature,*

$$K = 0,$$

*if and only if it is a developable.*  $\square$

We have also established that developables are characterized by the condition

$$\ddot{\rho} \ddot{\mathbf{a}} = 0$$

which is easily seen to be equivalent to the collinearity of the vectors  $\dot{\rho} \times \mathbf{a}$  and  $\dot{\mathbf{a}} \times \mathbf{a}$ . But the collinearity of these ve-

ctors is equivalent to the fact that the vector

$$\mathbf{r}_u \times \mathbf{r}_v = \dot{\mathbf{p}} \times \mathbf{a} + v (\dot{\mathbf{a}} \times \mathbf{a})$$

is, up to proportionality, independent of  $v$ , i.e. independent of  $v$  is the corresponding unit vector  $\mathbf{n}$ . This proves that *developables can be distinguished among all the ruled surfaces by the property that at all the points of each rectilinear generator such a surface has the same tangential plane*.  $\square$

For an arbitrary surface of revolution

$$\mathbf{r} = x(v) \cos u \cdot \mathbf{i} + x(v) \sin u \cdot \mathbf{j} + z(v) \mathbf{k}$$

we have

$$\mathbf{r}_u = -x(v) \sin u \cdot \mathbf{i} + x(v) \cos u \cdot \mathbf{j},$$

$$\mathbf{r}_v = x'(v) \cos u \cdot \mathbf{i} + x'(v) \sin u \cdot \mathbf{j} + z'(v) \mathbf{k}$$

and hence  $E = x(v)^2$ ,  $F = 0$ ,  $G = 1$  (we assume that  $x'(v)^2 + z'(v)^2 = 1$ ; see Lecture 25). Therefore

$$\mathbf{r}_u \times \mathbf{r}_v = x(v) z'(v) \cos u \cdot \mathbf{i} + x(v) z'(v) \sin u \cdot \mathbf{j} - x(v) x'(v) \mathbf{k},$$

$$\mathbf{n} = z'(v) \cos u \cdot \mathbf{i} + z'(v) \sin u \cdot \mathbf{j} - x'(v) \mathbf{k},$$

$$\mathbf{r}_{uu} = -x(v) \cos u \cdot \mathbf{i} - x(v) \sin u \cdot \mathbf{j},$$

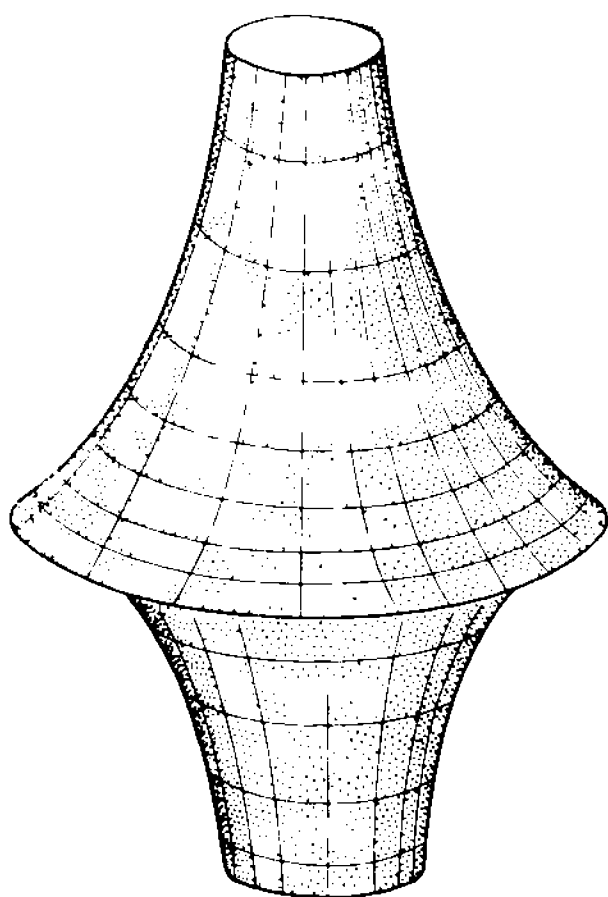
$$\mathbf{r}_{uv} = -x'(v) \sin u \cdot \mathbf{i} + x'(v) \cos u \cdot \mathbf{j},$$

$$\mathbf{r}_{vv} = x''(v) \cos u \cdot \mathbf{i} + x''(v) \sin u \cdot \mathbf{j} + z''(v) \mathbf{k};$$

$$\mathbf{L} = \mathbf{r}_{uu} \mathbf{n} = -x(v) z'(v), \quad \mathbf{M} = \mathbf{r}_{uv} \mathbf{n} = 0,$$

$$N = x''(v) z'(v) - z''(v) x'(v) = - \begin{vmatrix} x'(v) & z'(v) \\ x''(v) & z''(v) \end{vmatrix},$$

$$\frac{LN - M^2}{EG - F^2} = \frac{z'(v)}{x(v)} \begin{vmatrix} x'(v) & z'(v) \\ x''(v) & z''(v) \end{vmatrix}.$$



A pseudosphere

This proves that for a surface of revolution

$$K = \frac{z'(\nu)}{x(\nu)} \begin{vmatrix} x'(\nu) & z'(\nu) \\ x''(\nu) & z''(\nu) \end{vmatrix}.$$

**Example 1.** For a sphere of radius  $R$  we have

$$x(\nu) = R \cos \frac{\nu}{R}, \quad z(\nu) = R \sin \frac{\nu}{R},$$

and therefore

$$x'(\nu) = -\sin \frac{\nu}{R}, \quad z'(\nu) = \cos \frac{\nu}{R},$$

$$x'' = -\frac{1}{R} \cos \frac{\nu}{R}, \quad z''(\nu) = -\frac{1}{R} \sin \frac{\nu}{R},$$

$$K = \frac{z'(\nu)}{x(\nu)} \begin{vmatrix} x'(\nu) & z'(\nu) \\ x''(\nu) & z''(\nu) \end{vmatrix} = \frac{1}{R^2}.$$

Thus the total curvature of a sphere of radius  $R$  is constant and equal to  $1/R^2$ .  $\square$

The result is intuitively obvious.

The following example is more interesting.

**Example 2.** A surface of revolution with profile

$$x(\nu) = R \sin \nu, \quad z(\nu) = R \left( \ln \tan \frac{\nu}{2} + \cos \nu \right), \quad 0 < \nu < \frac{\pi}{2}$$

(it is the so-called *tractrix*) is termed a *pseudosphere*. For this surface

$$x'(\nu) = R \cos \nu, \quad z'(\nu) = \frac{R}{\sin \nu} - R \sin \nu = R \frac{\cos^2 \nu}{\sin \nu}$$

and hence

$$x'(\nu)^2 + z'(\nu)^2 = R^2 \cot^2 \nu.$$

Since  $x'(\nu)^2 + z'(\nu)^2 = 1$ , the general formula obtained above is not applicable directly and it is necessary to first pass to the natural parameter of the profile.

We have

$$s = -R \int_{\frac{\pi}{2}}^{\nu} \cot \nu \, d\nu = -R \ln \sin \nu$$



and hence

$$\sin v = e^{-\frac{s}{R}}, \quad \cos v = \sqrt{1 - e^{-2\frac{s}{R}}},$$

$$\tan \frac{v}{2} = e^{\frac{s}{R}} - \sqrt{e^{2\frac{s}{R}} - 1}.$$

Thus in terms of the natural parameter (which is again denoted by  $v$ ) the tractrix will be given by the functions

$$x(v) = Re^{-\frac{v}{R}},$$

$$z(v) = R \ln \left( e^{\frac{v}{R}} - \sqrt{e^{2\frac{v}{R}} - 1} \right) + R \sqrt{1 - e^{-2\frac{v}{R}}}.$$

We calculate:

$$x'(v) = -e^{-\frac{v}{R}}, \quad z'(v) = -\sqrt{1 - e^{-2\frac{v}{R}}},$$

$$x''(v) = \frac{1}{R}e^{-\frac{v}{R}}, \quad z''(v) = -\frac{e^{-2\frac{v}{R}}}{R\sqrt{1 - e^{-2\frac{v}{R}}}},$$

$$\begin{vmatrix} x'(v) & z'(v) \\ x''(v) & z''(v) \end{vmatrix} = -\frac{e^{-\frac{v}{R}}}{R\sqrt{1 - e^{-2\frac{v}{R}}}},$$

$$\frac{z'(v)}{x(v)} \begin{vmatrix} x'(v) & z'(v) \\ x''(v) & z''(v) \end{vmatrix} = -\frac{1}{R^2}.$$

Thus

$$K = -\frac{1}{R^2},$$

so that *the total curvature of a pseudosphere is constant and equal to  $-\frac{1}{R^2}$* .  $\square$

We see that in regard to total curvature the pseudosphere differs from the sphere only in the sense of curvature. This accounts for the term “pseudosphere”.

**Example 3.** For the catenoid

$$\begin{aligned}x(v) &= \operatorname{ch} v, \quad z(v) = v, \\x'(v) &= \operatorname{sh} v, \quad z'(v) = 1, \\x'(v)^2 + z'(v)^2 &= \operatorname{ch}^2 v,\end{aligned}$$

and therefore we must again pass to the natural parameter

$$s = \int_0^v \operatorname{ch} v \, dv = \operatorname{sh} v.$$

Again denoting this parameter by  $v$  we obtain the functions

$$x(v) = \sqrt{1+v^2}, \quad z(v) = \ln(v + \sqrt{1+v^2}).$$

Therefore

$$\begin{aligned}x'(v) &= \frac{v}{\sqrt{1+v^2}}, \quad z'(v) = \frac{1}{\sqrt{1+v^2}}, \\x''(v) &= \frac{1}{(1+v^2)^{3/2}}, \quad z''(v) = -\frac{v}{(1+v^2)^{3/2}}, \\ \left| \begin{array}{cc} x'(v) & z'(v) \\ x''(v) & z''(v) \end{array} \right| &= -\frac{1}{1+v^2},\end{aligned}$$

and hence

$$K = -\frac{1}{(1+v^2)^2}.$$

It is interesting to compare the curvature of the catenoid with that of the helicoid isometric with it.

For the helicoid we have equation (11) with

$$\rho(u) = u\mathbf{k}, \quad \mathbf{a}(u) = \cos u \cdot \mathbf{i} + \sin u \cdot \mathbf{j}.$$

Therefore

$$\begin{aligned}\dot{\rho} &= \mathbf{k}, \quad \dot{\mathbf{a}} = -\sin u \cdot \mathbf{i} + \cos u \cdot \mathbf{j}, \\E &= 1 + 2v\dot{\rho}\ddot{\mathbf{a}} + v^2\dot{\mathbf{a}}^2 = 1 + v^2, \\F &= \dot{\rho}\mathbf{a} = 0, \quad G = 1, \\EG - F^2 &= 1 + v^2, \\\dot{\rho}\mathbf{a}\ddot{\mathbf{a}} &= \begin{vmatrix} 0 & 0 & 1 \\ \cos u & \sin u & 1 \\ -\sin u & \cos u & 0 \end{vmatrix} = 1,\end{aligned}$$

and hence

$$K = -\frac{1}{(1+v^2)^2},$$

We have obtained the same result as that for the catenoid! This means that *the total curvatures coincide at the corresponding points when the catenoid is bent into the helicoid.*  $\square$

What happens to the mean curvature?

For the catenoid  $E = 1 + v^2$ ,  $F = 0$ ,  $G = 1$ . In addition

$$L = -x(v)z'(v) = -1, \quad M = 0,$$

$$N = -\begin{vmatrix} x'(v) & z'(v) \\ x''(v) & z''(v) \end{vmatrix} = \frac{1}{1+v^2},$$

and therefore

$$EN + GL - 2FM = 0,$$

i.e.

$$H = 0.$$

Thus *the mean curvature of the catenoid is equal to zero.*  $\square$

For the helicoid, on the other hand,

$$\dot{\rho} \times \mathbf{a} = \sin u \cdot \mathbf{i} - \cos u \cdot \mathbf{j}, \quad \dot{\mathbf{a}} \times \mathbf{a} = -\mathbf{k},$$

$$\dot{\rho} = 0, \quad \ddot{\mathbf{a}} = -\cos u \cdot \mathbf{i} - \sin u \cdot \mathbf{j},$$

$$(\ddot{\rho} + v\ddot{\mathbf{a}})(\dot{\rho} \times \mathbf{a} + v(\dot{\mathbf{a}} \times \mathbf{a})) = 0$$

and in addition, as we have already seen,

$$E = 1 + v^2, \quad F = 0, \quad G = 1,$$

$$EG - F^2 = 1 + v^2, \quad \dot{\rho} \mathbf{a} \dot{\mathbf{a}} = 1.$$

Therefore

$$L = 0, \quad M = \frac{1}{\sqrt{1+v^2}}, \quad N = 0,$$

and hence

$$EN + LG - 2FM = 0,$$

i.e.

$$H = 0.$$

Thus *the mean curvature of the helicoid is also equal to zero.*  $\square$

The example of catenoid and helicoid suggests that total and mean curvatures are preserved under bending (isometry). It turns out that *this hypothesis is true* for total curvature (and we shall show this in our next lecture) whereas for mean curvature *it is false*. Indeed, for a plane the mean curvature is equal to zero while for a circular cylinder of radius  $R$  developable into a plane it is obviously equal to  $1/2R$ .

The reasons why the catenoid and helicoid have turned out to have equal mean curvatures are deep and interesting but we are deprived of the possibility of discussing them here.

# Lecture 27

---

*Weingarten's derivation formulas • Coefficients of connection • The Gauss theorem • The necessary and sufficient conditions of isometry*

---

For the moving basis  $\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}$  of an arbitrary surface

$$(1) \quad \mathbf{r} = \mathbf{r}(u, v)$$

formulas can be written, similar to Frenet's formulas for curves, that yield an expansion of the derivatives

$$\mathbf{r}_{uu}, \mathbf{r}_{uv}, \mathbf{r}_{vv}, \mathbf{n}_u, \mathbf{n}_v$$

of the vectors of the moving basis with respect to that same basis.

Since  $\mathbf{n}^2 = 1$  and hence  $\mathbf{n}\mathbf{n}_u = 0$  and  $\mathbf{n}\mathbf{n}_v = 0$ , the vectors  $\mathbf{n}_u$  and  $\mathbf{n}_v$  are expanded only with respect to the vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ , so that

$$\mathbf{n}_u = \alpha \mathbf{r}_u + \beta \mathbf{r}_v,$$

$$\mathbf{n}_v = \alpha_1 \mathbf{r}_u + \beta_1 \mathbf{r}_v.$$

Multiplying the first of these formulas by  $\mathbf{r}_u$  and  $\mathbf{r}_v$  we obtain two relations:

$$-L = \mathbf{r}_u \mathbf{n}_u = \alpha \mathbf{r}_u^2 + \beta \mathbf{r}_u \mathbf{r}_v = \alpha E + \beta F,$$

$$-M = \mathbf{r}_v \mathbf{n}_u = \alpha \mathbf{r}_u \mathbf{r}_v + \beta \mathbf{r}_v^2 = \alpha F + \beta G,$$

from which it follows that

$$\alpha = \frac{FM - GL}{EG - F^2}, \quad \beta = \frac{FL - EM}{EG - F^2}.$$

Similarly calculated are the coefficients of the second formula:

$$\alpha_1 = \frac{FN - GM}{EG - F^2}, \quad \beta_1 = \frac{FM - EN}{EG - F^2}.$$

Further, since by definition

$$\mathbf{r}_{uu}\mathbf{n} = L, \quad \mathbf{r}_{uv}\mathbf{n} = M, \quad \mathbf{r}_{vv}\mathbf{n} = N$$

and since by the hypothesis  $\mathbf{r}_u\mathbf{n} = 0$ ,  $\mathbf{r}_v\mathbf{n} = 0$ , the coefficients of  $\mathbf{n}$  in the expansions of the vectors  $\mathbf{r}_{uu}$ ,  $\mathbf{r}_{uv}$ ,  $\mathbf{r}_{vv}$  with respect to the basis  $\mathbf{r}_u$ ,  $\mathbf{r}_v$ ,  $\mathbf{n}$  are equal to  $L$ ,  $M$ ,  $N$  respectively.

We thus have

$$\begin{aligned} \mathbf{r}_{uu} &= \Gamma_{11}^1 \mathbf{r}_u + \Gamma_{11}^2 \mathbf{r}_v + L\mathbf{n}, \\ \mathbf{r}_{uv} &= \Gamma_{12}^1 \mathbf{r}_u + \Gamma_{12}^2 \mathbf{r}_v + M\mathbf{n}, \\ \mathbf{r}_{vv} &= \Gamma_{22}^1 \mathbf{r}_u + \Gamma_{22}^2 \mathbf{r}_v + N\mathbf{n}, \\ \mathbf{n}_u &= \frac{FM - GL}{EG - F^2} \mathbf{r}_u + \frac{FL - EM}{EG - F^2} \mathbf{r}_v, \\ \mathbf{n}_v &= \frac{FN - GM}{EG - F^2} \mathbf{r}_u + \frac{FM - EN}{EG - F^2} \mathbf{r}_v, \end{aligned} \quad (2)$$

where  $\Gamma_{ij}^k$ ,  $i, j, k = 1, 2$ , are some functions of  $u$  and  $v$ . Formerly these functions were designated by the symbols

$$\left\{ \begin{matrix} i & j \\ & k \end{matrix} \right\}$$

and called *Christoffel symbols*. But now they are usually called *connection coefficients*.

Formulas (2) are called *Weingarten's derivation formulas*.

To compute connection coefficients  $\Gamma_{ij}^k$  we first find the six products of vectors  $\mathbf{r}_{uu}$ ,  $\mathbf{r}_{uv}$ ,  $\mathbf{r}_{vv}$  by vectors  $\mathbf{r}_u$  and  $\mathbf{r}_v$ .

Since  $\mathbf{r}_u^2 = E$ , we have  $2\mathbf{r}_{uu}\mathbf{r}_u = E_u$  and  $2\mathbf{r}_{uv}\mathbf{r}_u = E_v$ , i.e.

$$\mathbf{r}_{uu}\mathbf{r}_u = \frac{1}{2} E_u \quad \text{and} \quad \mathbf{r}_{uv}\mathbf{r}_u = \frac{1}{2} E_v.$$

Similarly, since  $\mathbf{r}_v^2 = G$ , we have

$$\mathbf{r}_{uv}\mathbf{r}_v = \frac{1}{2} G_u \quad \text{and} \quad \mathbf{r}_{vv}\mathbf{r}_v = \frac{1}{2} G_v.$$

Besides, since  $\mathbf{r}_u \mathbf{r}_v = F$ , we have  $\mathbf{r}_{uu} \mathbf{r}_v + \mathbf{r}_u \mathbf{r}_{uv} = F_u$  and  $\mathbf{r}_{uv} \mathbf{r}_v + \mathbf{r}_u \mathbf{r}_{vv} = F_v$ , from which it follows that

$$\mathbf{r}_{uu} \mathbf{r}_v = F_u - \frac{1}{2} E_v \quad \text{and} \quad \mathbf{r}_{vv} \mathbf{r}_u = F_v - \frac{1}{2} G_u.$$

Now multiplying the first three of the formulas (2) by  $\mathbf{r}_u$  and  $\mathbf{r}_v$  we obtain six relations:

$$\begin{cases} E\Gamma_{11}^1 + F\Gamma_{11}^2 = \frac{1}{2} E_u, \\ E\Gamma_{11}^1 + G\Gamma_{11}^2 = F_u - \frac{1}{2} E_v, \\ E\Gamma_{12}^1 + FG_{12}^2 = \frac{1}{2} E_v, \\ F\Gamma_{12}^1 + G\Gamma_{12}^2 = \frac{1}{2} G_u, \\ E\Gamma_{22}^1 + F\Gamma_{22}^2 = F_v - \frac{1}{2} G_u, \\ F\Gamma_{22}^1 + G\Gamma_{22}^2 = \frac{1}{2} G_v, \end{cases}$$

from which it is easy to find the coefficients  $\Gamma_{ij}^k$ .

(The equations are uniquely solvable since the determinant  $EG - F^2$  of every pair of equations is nonzero.)

We see that the *connection coefficients*  $\Gamma_{ij}^k$  can be expressed in terms of the coefficients of the first quadratic form and of their derivatives. Hence they remain unaffected under bendings (isometries) of a surface.  $\square$

We shall not need explicit expressions for coefficients  $\Gamma_{ij}^k$  in terms of the coefficients of the first quadratic form, and so we shall not write them out.

The coefficients of derivation formulas are connected by three relations resulting from calculating partial derivatives  $\mathbf{r}_{uuv}$ ,  $\mathbf{r}_{uvv}$ , and  $\mathbf{n}_{uv}$  in two different ways using these formulas. One of these relations was found by Gauss and the other two by Peterson, Codazzi and Mainardi. We shall consider only Gauss' relation which we shall obtain by calculating the coefficient of  $\mathbf{r}_v$  in the expansion of the partial derivative  $\mathbf{r}_{uuv}$  with respect to the vectors  $\mathbf{r}_u$ ,  $\mathbf{r}_v$ , and  $\mathbf{n}$ .

In this calculation we shall only follow the coefficient of  $\mathbf{r}_v$  and only those of its terms which depend on the coefficients of the second quadratic form. All the other terms will be replaced by dots.

We have

$$\begin{aligned}\mathbf{r}_{uu v} &= (\mathbf{r}_{uu})_v = (\Gamma_{11}^1 \mathbf{r}_u + \Gamma_{11}^2 \mathbf{r}_v + L \mathbf{n})_v = \\ &= \dots + \Gamma_{11}^1 \mathbf{r}_{uv} + \dots + \Gamma_{11}^2 \mathbf{r}_{vv} + \dots + L \mathbf{n}_v = \\ &= \dots + \Gamma_{11}^1 (\dots) + \dots + \Gamma_{11}^2 (\dots) + \dots \\ &\quad \dots + L \left( \dots + \frac{FM - EN}{EG - F^2} \mathbf{r}_v \right) = \\ &= \left( L \frac{FM - EN}{EG - F^2} + \dots \right) \mathbf{r}_v + \dots\end{aligned}$$

Similarly

$$\begin{aligned}\mathbf{r}_{uuv} &= (\mathbf{r}_{uv})_u = (\Gamma_{12}^1 \mathbf{r}_u + \Gamma_{22}^2 \mathbf{r}_v + M \mathbf{n})_u = \\ &= \left( M \frac{FL - EM}{EG - F^2} + \dots \right) \mathbf{r}_v + \dots\end{aligned}$$

Hence

$$L \frac{FM - EN}{EG - F^2} = M \frac{FL - EM}{EG - F^2} + \dots,$$

where dots denote terms depending only on the coefficient of the first quadratic form. But

$$M \frac{FL - EM}{EG - F^2} - L \frac{FM - EN}{EG - F^2} = E \frac{LN - M^2}{EG - F^2} = EK.$$

Since  $E \neq 0$  (form  $I$  is positive definite), this proves that *the total curvature  $K$  of a surface is expressible in terms of the coefficients of the first quadratic form (and of their derivatives)*. It follows that the curvature  $K$  remains unaffected under bendings. This result deserves to be distinguished as a theorem.

**Theorem 1 (the Gauss theorem).** *The total (Gaussian) curvature of a surface remains unaffected under bendings (isometries), i.e. isometric surfaces have the same curvature at points corresponding to each other.*  $\square$

Gauss was so delighted with the theorem that he called it *theorema egregium*, which means a “brilliant theorem” in Latin. From Theorem 1 it follows in particular that no arbit-



rarily small part of a sphere can be bent into a plane. Therefore no map gives an absolutely faithful representation of the Earth's surface.

An explicit expression for curvature  $K$  in terms of the coefficients  $E$ ,  $F$ , and  $G$  of the first quadratic form is

$$(3) \quad K = -\frac{1}{4(EG-F^2)^2} \begin{vmatrix} E & E_u & E_v \\ F & F_u & F_v \\ G & G_u & G_v \end{vmatrix} - \frac{1}{2\sqrt{EG-F^2}} \left\{ \left( \frac{E_v-F_u}{\sqrt{EG-F^2}} \right)_v - \left( \frac{F_v-G_u}{\sqrt{EG-F^2}} \right)_u \right\}.$$

The other two relations, obtained from differentiating the derivation formulas (and usually called the *Peterson-Codazzi formulas*) are of the form

$$2(EG-F^2)(L_v-M_u) - (EN+GL-2FM)(E_v-F_u) + \begin{vmatrix} E & E_u & L \\ F & F_u & M \\ G & G_u & N \end{vmatrix} = 0,$$

(4)

$$2(EG-F^2)(M_v-N_u) - (EN+GL-2FM)(F_v-G_u) + \begin{vmatrix} E & E_v & L \\ F & F_v & M \\ G & G_v & N \end{vmatrix} = 0.$$

To prove these formulas all one needs is patience and carefulness.

The Gauss theorem states that the equality of total curvatures is a necessary condition for the isometry of two surfaces. At the same time, although this condition is by no means sufficient, it is so strong that using it sufficient conditions can be obtained without difficulty. We shall not expound this question and only consider the most important special case of the corresponding theorem.

Let

$$\Delta_1 K = \frac{EK_v^2 - 2FK_uK_v + GK_u^2}{EG-F^2}.$$

(It is Beltrami's first differential parameter of the function  $K$  calculated in "curvilinear" coordinates  $u$  and  $v$ ). If the two functions  $K$  and  $\Delta_1 K$  of  $u$  and  $v$  are *functionally independent*, i.e. if their Jacobian

$$\begin{vmatrix} \frac{\partial K}{\partial u} & \frac{\partial K}{\partial v} \\ \frac{\partial \Delta_1 K}{\partial u} & \frac{\partial \Delta_1 K}{\partial v} \end{vmatrix}$$

is nonzero, then they may be taken as new local coordinates on surface (1). We call these coordinates *Gaussian* coordinates. A direct calculation shows that any diffeomorphism of a surface preserving the function  $K$  (in particular any isometry) leaves the function  $\Delta_1 K$  invariant too. In particular every isometry is therefore a mapping defined by equating Gaussian coordinates. This means that the following theorem is true.

**Theorem 2.** *Two surfaces which have Gaussian coordinates defined on them are isometric if and only if in these coordinates their first quadratic forms coincide.  $\square$*

Thus, to determine whether or not two surfaces are isometric it is necessary to introduce (if possible) Gaussian coordinates and calculate in these coordinates the first quadratic forms of the surfaces. If the forms coincide, the surfaces are isometric, but if they are different, the surfaces are not isometric.

Theorem 2 gives no answer when  $K$  and  $\Delta_1 K$  are functionally dependent, for example when  $\Delta_1 K = 0$  (which occurs, as can be easily figured out, if and only if  $K = \text{const}$ ). In this extreme case it can be shown, however, that the condition of Theorem 1 proves to be sufficient, i.e. *two surfaces of constant total curvature are isometric if and only if they have the same curvature*. In other words, *any surface of constant total curvature  $K$  is isometric with a sphere of radius*

$R = \frac{k}{\sqrt{K}}$  if  $K > 0$ , with a plane if  $K = 0$ , and with a pseu-

dosphere with parameter  $R = \frac{1}{\sqrt{-K}}$  if  $K < 0$ . The proof

consists in constructing explicitly coordinates  $u, v$  in which the first quadratic form coincides with the first quadratic form of a sphere, a plane, and a pseudosphere respectively. Unfortunately we have no time to spare.

## SUBJECT INDEX

- Alternation, 67
- Anti-isomorphism of algebras, 147
- Basis conjugate to a basis, 38
- Beltrami's first differential parameter, 242
- Catenary, 287
- Catenoid, 287
- Christoffel brackets, 311
- Coefficients of connection, 311
- Complement, orthogonal, 181
- Cone,  $k$ -fold, 128
- Coordinates, contravariant, 189
  - covariant, 189
  - Plücker, 97
- Covector, 36
- Curvature, radius of, 257
- Curve, catenary, 287
  - curvature of, 255, 258
  - curvatures of, 265
  - in a Euclidean (affine) space, 243
  - mean curvature of, 299
  - nonparametric, 246
  - of the general type, 257, 258, 263
  - regular, 245
  - relative curvature of, 251
  - smooth, 244
- Cylinder, 272
  - $k$ -fold, 128
- Darboux's differential parameter, 242
- Developables, 289
- Diffeomorphism, 270
- Eigenvalue, 151
  - algebraic multiplicity of, 153
  - geometrical multiplicity of, 151
- Eigenvalues, minimax property of, 204
- Eigenvector, 151
- Factor space of a vector modulo a subspace, 32
- Form (s), normal, 116
  - quadratic, 112
    - law of inertia for, 122
    - positive definite, 124
    - of a surface, first, 279
    - second, 296
- Fredholm alternative, 60
- Frenet's formulas, 257, 259, 265
- Frenet's moving basis, 257, 258, 264

- Function, gradient of, 223
  - graph of, 224
  - harmonic, 241
- Functional, bilinear, 48
  - matrix of, 49
  - matrix rank of, 108
  - mixed, 52
- linear, 36
  - tensor product of, 50
- multilinear, 62
  - degree of, 62
  - rank of, 65
- rank space of, 62
- quadratic, 110
  - positive definite, 124
- sesquilinear, 179
- skew-symmetric, 73
  - essential coefficients of, 82
- symmetric, 106
- Gauss theorem, 313
- Grassman algebra, 77
- Grassman manifold, 105
- Group, symmetric, 66
  - unitary, 213
- Hamilton-Cayley theorem, 171
- Hamilton's symbolic vector field, 236
- Helicoid, 287
- Helix, circular, 259
- Hyperplane, 98
  - tangential, 253
- Hypersurface, of an affine space, 135
  - central, 133
  - noncentral second degree, 133
  - oval second degree, 135
  - of a projective space, 129
  - second degree, 134, 137
  - singular point of, 228
  - smooth (regular), 223
- Index, inertial, 123
  - negative, 123
  - positive, 123
- lowering of, 189
- raising of, 189
- Indicatrix of Dupin, 296
- Jacobi theorem, 121
- Jordan cell, 167
- Jordan normal form, 167
- Kronecker delta symbol, 42
- Kronecker-Capelli theorem, 25
- Lagrange algorithm, 114
- Lagrange theorem, 114
- Length of a continuous curve, 253
- Manifold, linear, 99
- Mapping, linear, 33
- Matrix, of the coefficients, 25
  - augmented, 25
  - positive definite, 124
  - rank of, 20
  - unitriangular, 118
- Minor, 20
- Minor, principal, 119
- Minor determinant, 20
- Multivector, of degree  $p$ , 86
- Net, coordinate, 156
- Nilpotency of a (matrix) operator, degree of, 158
- Operator, Laplacian, 241
  - linear, 140
    - adjoint, 191
    - complexification of, 171
    - cyclic, 159
    - diagonalizable, 156
    - Hermitian, 192

- Operator, linear (cont.)
  - identity, 141
  - invertible, 144
  - isometric, 212
  - matrix of, 143
  - nilpotent, 158
  - nonnegative, 209
  - nonsingular, 144
  - normal, 205
  - orthogonal, 212
  - orthogonally diagonalizable, 198
  - positive, 209
  - restriction of, 149
  - scalar, 142
  - self-adjoint, 193
  - singular, 144
  - skew-Hermitian, 193
  - skew-symmetric, 193
  - spectrum of, 157
  - symmetrical, 192
  - unitary, 217
  - zero, 141
- Pairing, 39
- Paraboloid, elliptical, 137
  - hyperbolic, 137
- Permutation, 66
  - even, 66
  - odd, 66
- Peterson-Codazzi formulas, 314
- Plane, normal, 263
  - osculating, 263
  - of a projective space,  $r$ -dimensional, 102
  - parametric vector equation of, 99
  - $p$ -dimensional, 98
  - rectifying, 263
- Plücker relations, 95
- Point of self-intersection, 249
- Polynomial, characteristic, 153
  - in an operator, 168
- Product, external, 75
- Pseudosphere, 305
- Rank, external, 88
- Root subspace, 163
- Root vector, 163
- Rotation 215
  - generalized, 215
- Set, annulet of, 44
  - linear span of, 12
  - rank of, 17
  - span of, 12
- Simple spectrum linear operator, 157
- Space, pseudo-Euclidean, 113
  - conjugate to a vector space, 36
- Spaces, direct sum of, 35
- Straight line, 98
- Submatrix, principal, 119
- Subspace, 11
  - belonging to an eigenvalue, 151
  - direction  $p$ -vector of, 97
  - invariant, 149
  - trivial, 12
  - zero, 12
- Subspaces, complementary, 30
  - direct sum of, 28
  - sum of, 14
- Surface, elliptical point of, 29
  - hyperbolic point of, 298
  - of binormals, 275
  - of principal normals, 275
  - of tangents, 275
  - parabolic point of, 298
  - principal curvatures of, 299
  - regular, 270
  - ruled, 274

- (Surface cont.)
  - support of, 270
  - total curvature of, 299
- Surfaces, developable, 289
  - isometric, 283
  - isometry of, 283
- Sylvester's criterion, 125
- System of solutions, fundamental, 45
- Tangent vector, 245, 250
- Tensor, coefficients of, 55
  - contraction of, 60
  - $(p, q)$ -, 54
- Tensor transformation law, 56
- Torsion, 259
- Trace, 61
- Tractrix, 305
- Transformation, affine,
  - centroaffine, 215
  - orthogonal, 214
  - unitary, 214
- Translation, 215
  - parallel, 215
- Vector, binormal, 258
  - of the principal normal to a curve, 258
  - to a surface, normal, 291
- Vectors, congruent modulo a subspace, 31
  - tensor product of, 50
- Vector field, 227
  - divergence of, 234
  - gradient, 230
  - irrotational, 230
  - potential, 230
    - rotation of, 231
  - singular point of, 228
- Vector potential, 234
- Vector space, 11
  - complexification of, 170
  - tensor algebra of, 59
- Vector spaces, coimage of a homomorphism of, 34
  - dual, 39
  - epimorphism of, 33
  - homomorphism of, 33
  - image of a homomorphism of, 33
  - kernel of a homomorphism of, 33
    - monomorphism of, 33
    - morphism of, 33
- Weingarten's derivation formulas, 311

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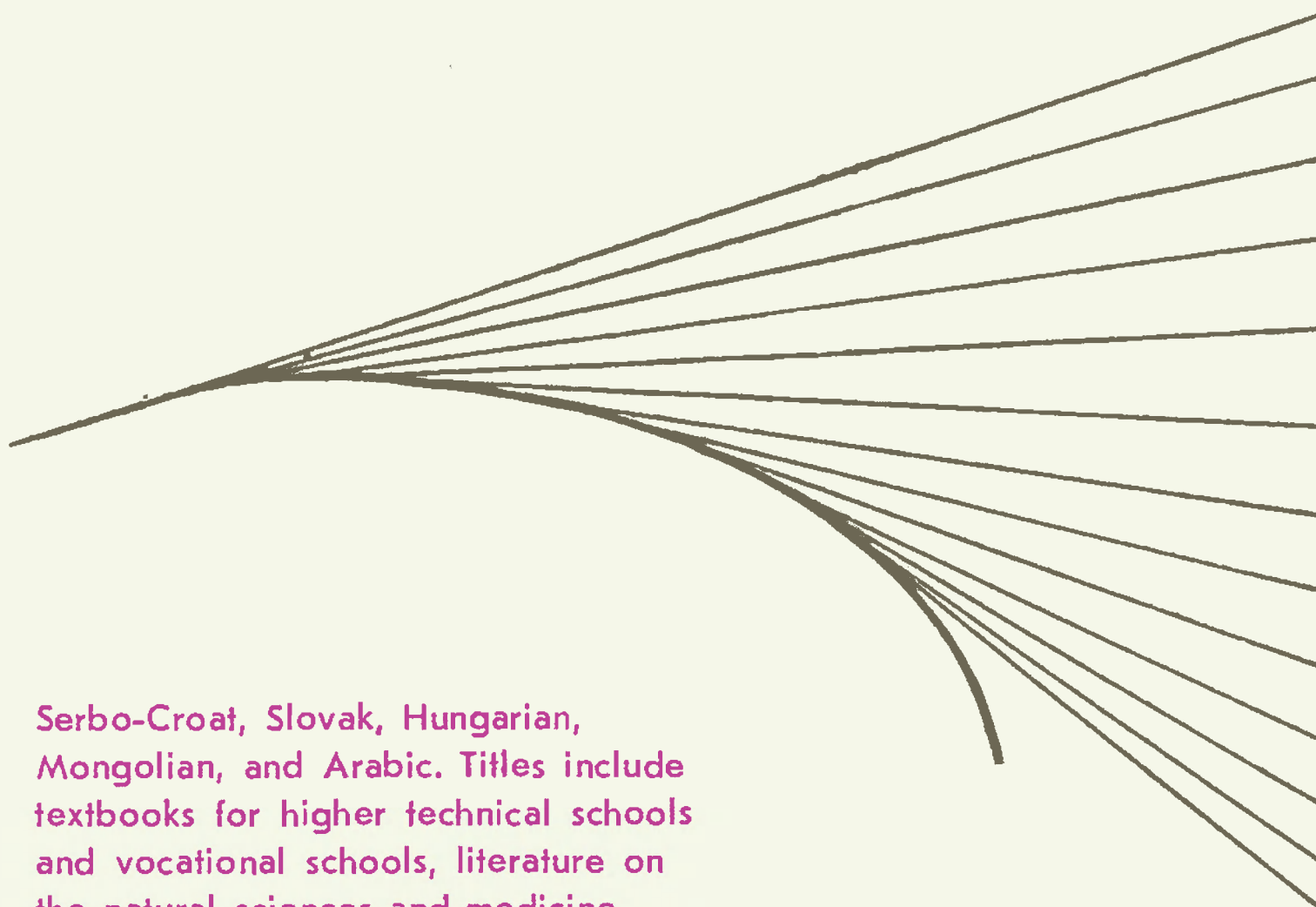


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